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Solving Second-Order Differential Equations with Lie Symmetries

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Abstract. Lie's theory for solving second-order quasilinear differential equations based on its symmetries is discussed in detail. Great importance is attached to *constructive* procedures that may be applied for designing solution algorithms. To this end Lie's original theory is supplemented by various results that have been obtained after his death one hundred years ago. This is true above all of Janet's theory for systems of linear partial differential equations and of Loewy's theory for decomposing linear differential equations into components of lowest order. These results allow it to formulate the equivalence problems connected with Lie symmetries more precisely. In particular, to determine the function field in which the transformation functions act is considered as part of the problem. The equation that originally has to be solved determines the *base field*, i.e. the smallest field containing its coefficients. Any other field occurring later on in the solution procedure is an extension of the base field and is determined explicitly. An equation with symmetries may be solved in closed form *algorithmically* if it may be transformed into a canonical form corresponding to its symmetry type by a transformation that is *Liouvillian* over the base field. For each symmetry type a solution algorithm is described, it is illustrated by several examples. Computer algebra software on top of the type system ALLTYPES has been made available in order to make it easier to apply these algorithms to concrete problems.

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"...Es geht mir vorwärts, aber langsam, und es kostet unendliche und äußerst langweilige Rechnungen..." Sophus Lie, Letter to A. Mayer of April 4, 1874*

1. Introduction

Solving equations has been one of the most important driving forces in the history of mathematics. Particularly well known is the problem of solving algebraic equations of orders higher than four and its eventual solution by Lagrange, Ruffini, Abel and above all Galois. The complete answer given by the latter author provided not only the solution for the problem at hand but also established a new field in

* Translation by the author: "...I proceed, but slowly, and it takes me infinite, extremely boring calculations...".