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Decomposing and Solving Quasilinear Second-Order Differential Equations

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Abstract

Decompositions of *linear* ordinary differential equations (ode's) into components of lower order have successfully been employed for determining its solutions. Here this method is generalized to certain classes of *quasilinear* equations of second order, i.e. equations that are linear w.r.t. the second derivative, and rational otherwise. Often it leads to simple expressions for the general solution that hardly can be obtained otherwise, i.e. it is a genuine extension of Lie's symmetry analysis. Due to its efficiency it is suggested that it is applied always as a first step in an ode solver.

1 Introduction

The goal of this article is to find closed form solutions of quasilinear second-order differential equations. i.e. finite expressions in terms of known functions like e.g. elementary or Liouvillian functions that annihilate the given differential equation upon substitution; they exist only in exceptional cases. For linear equations a rather detailed solution scheme based on factorization and decomposition is available [8].

For nonlinear equations the most important solution procedure is based on Lie's symmetry analysis; in Appendix E of [6] it has been shown that many solutions of non-linear second order equations listed in Kamke's collection [5] are based on the existence of nontrivial Lie symmetries. Yet the situation is not completely satisfactory. There are second-order equations with a large group of Lie symmetries that may not be utilized for solving it as may be seen in several examples given below. On the other hand, there are equations without a nontrivial group of Lie symmetries but with a fairly simple closed-form solution, e.g. equation (1.5) of [6], page 6. In Section 3 it is shown how its general solution may be obtained by a proper decomposition.

This situation suggests trying an approach based on factorization and decomposition as it has been applied quite successfully to linear equations. In this article this proceeding will be applied to second-order quasilinear equations of the form

$$y'' + \sum_{k=0}^{K} c_k(x,y)y'^k = 0 \text{ with } c_k \in \mathbb{Q}(x,y), \;\; K \in \mathbb{N} \text{ and } K \geq 1; \tag{1}$$

throughout this article $y' \equiv \frac{dy}{dx}$ and $D \equiv \frac{d}{dx}$. For $K = 3$ it is called Lie's equation [6], page 219. The general solution of any second-order ode contains two independent constants. If it is possible to construct a first-order component from the given second-order equation it may be possible to determine a special solution depending on a single constant, or even the general solution as it is the case for linear equations.