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Solving Third Order Differential Equations with Maximal Symmetry Group

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Abstract

The largest group of Lie symmetries that a third-order ordinary differential equation (ode) may allow has seven parameters. Equations sharing this property belong to a single equivalence class with a canonical representative $v'''(u) = 0$. Due to this simple canonical form, any equation belonging to this equivalence class may be identified in terms of certain constraints for its coefficients. Furthermore a set of equations for the transformation functions to canonical form may be set up for which large classes of solutions may be determined *algorithmically*. Based on these steps a solution algorithm is described for any equation with this symmetry type which resembles a similar scheme for second order equations with projective symmetry group.

AMS Subject Classifications: 34A05, 34A25, 34B30, 68Q40.

Key Words: Differential equations, Lie symmetries, equivalence problems, computer algebra.

1. Introduction

Solution algorithms based on the Lie symmetries that an ordinary differential equation (ode) may admit have mainly been developed for second-order equations, originally by Lie himself [5], [6], more recent treatments of the subject may be found in the textbooks by Olver [8], Bluman and Kumei [1] and Ibragimov [2]. Special emphasis on constructive methods and solution algorithms are described in recent publications by the author [12] and [11]. The present article is a generalization of the latter publication for third order equations. In either case the respective symmetry type, i.e. the eight parameter projective group of the plane or a seven parameter group respectively, defines a single equivalence class with a particularly simple canonical form. In this article general non-canonical variables will always be denoted by x and $y \equiv y(x)$, they are called *actual variables*, whereas u and $v \equiv v(u)$ denote canonical variables, both sets are related by $u = \sigma(x, y)$ and $v = \rho(x, y)$. In these variables equations with maximal symmetry groups have the form $v'' = 0$ or $v''' = 0$ respectively. Due to the fact that the canonical forms are unique without any undetermined parameters of functions involved, equations contained in the respective equivalence class may be identified by their structure and certain relations obeyed by its coefficients. In either case, a complete solution scheme may be organized by the following three steps.