Generalized Loewy-Decomposition of D-Modules

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ABSTRACT

Starting from the well-known factorization of linear ordinary differential equations, we define the generalized Loewy decomposition for a $\mathcal{D}$-module. To this end, for any module I, overmodules $J \supseteq I$ are constructed. They subsume the conventional factorization as special cases. Furthermore, the new concept of the module of relative syzygies $Syz(I, J)$ is introduced. The invariance of this module and its solution space w.r.t. the set of generators is shown. We design an algorithm which constructs the Loewy-decomposition for finite-dimensional and some kinds of general $\mathcal{D}$-modules. These results are applied for solving various second- and third-order linear partial differential equations.

Categories and Subject Descriptors

I.1 [**Symbolic and Algebraic Manipulation**]: Applications

General Terms

Algorithms

Keywords

D-module, Loewy decomposition, Janet basis

Introduction

The concept of factorization of a linear ordinary differential equation (lode) originally goes back to Beke [1] and Schlesinger [21]. Loewy [14] extended it and introduced a unique decomposition of any lode into largest completely reducible factors, i. e. factors which are the least common multiple of irreducible right factors. Similar as in the algebraic case, if such a nontrivial decomposition may be found, the solution procedure is faciliated. Algorithms for factoring a lode have also been described by Schwarz [23] and, with improved complexity bounds, by Grigoriev [7]. A survey may be found in the book by Singer and van der Put [17].

Factoring linear partial differential equations (lpde's) is much more difficult. So far there has been no common agreement on what to understand by factoring lpde's in general. A first attempt to generalize the above theory by Li et al. [13], see also Tsarev [29], has been restricted to those lpde's which have a finite-dimensional solution space. This is achieved by a fairly straightforward extension of the factorization of lode's. Recently in [9] the problem of factoring a single lpde was studied. An algorithm was designed for factoring so-called separable lpde's, but the general factorization problem remained open.

Here an algebraic approach is suggested which subsumes the conventional factorizations and its corresponding decompositions as special cases. Any given linear differential equation is considered as the result of applying a differential operator to a differential indeterminate. This operator or, if a system of equations is involved, this set of operators, are considered as generators of a left $\mathcal{D}$-module over an appropriate ring of differential operators. Some background on $\mathcal{D}$-modules may be found e. g. in the book by Sabbah [20] or the article by Quadrat [18]. In our algebraic approach decomposing a $\mathcal{D}$-module means finding overmodules which describe various parts of the solution of the original problem. There are two possibilites for constructing these overmodules.

- A set of new generators is searched for such that the original module may be reduced to zero wrt. to them. This stands for the conventional factorization like factoring linear ode's [23], factoring linear pde's with a finite-dimensional solution space [13], or the factorizations that have been described in [9].

- It may be possible to construct new generators forming a Janet base of an overmodule in combination with the given ones, which are not necessarily of lower order.

In either case, the result is a set of operators generating an overmodule of the given one. The further proceeding depends on the result of this construction. It may occur that several over-modules have been obtained such that their intersection is identical to the given one. If this is true,

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solving the original problem is reduced to solving several, possibly simpler problems, each of which describes some part of the desired solution. In Loewy's terminology [14] such a module is called *completely reducible*.

If this case does not apply, for each over-module the module of *relative syzygies* is constructed as defined in Section 2 below. Then the same procedure is applied to it as for the originally given module. This process terminates until no further over-modules may be constructed. The result is the natural generalization of Loewy's decomposition of ordinary differential operators.

From this decomposition the solution of the originally given equation may be obtained iteratively. At first all homogeneous problems have to be solved. The solutions of the rightmost factors are already part of the solution of the full problem. In the next step the solutions of the module of relative syzygies are taken as inhomogeneity of the respective rightmost factor. Solving these problems yields additional parts of the solution of the full problem. This process is repeated until the last module of relative syzygies has been reached. If all equations that occur in this decomposition may be solved, the general solution of the original problem has been obtained or, if this is not true, at least some part of it.

In the first part of this article the algebraic background which is the basis of the above scheme is outlined. In Section 1 we show that the space of solutions of a module is determined by its class of isomorphisms (Proposition 1.1), up to an equivalence $\simeq_{\mathcal{D}}$ which is called $\mathcal{D}$-isomorphism.

In Section 2 we introduce the new concept of the module of relative syzygies $Syz(I,J)$ of two modules I and J with $I \subseteq J$. It extends the one given in [13] for finite-dimensional modules. It is shown that it is essentially invariant w.r.t. to the set of generators. We also show that for the space of solutions of $Syz(I,J)$ there holds $V_{Syz(I,J)} \simeq_{\mathcal{D}} V_I/V_J$ (Lemma 2.4), this provides a bijective correspondence between classes of isomorphisms of the factors I/J and classes of $\mathcal{D}$-isomorphisms of the solutions spaces $V_{Syz(I,J)}$ (Corollary 2.5). In addition we describe a procedure to calculate the module of relative syzygies. Finally, the relation $a_\tau(Syz(I,J)) = a_\tau(I) - a_\tau(J)$ (Theorem 2.7) is proved for the leading coefficients a_τ of the Hilbert-Kolchin polynomials; τ is the differential type of the module I, see [11] and [12].

In Section 3, at first we define a unique Loewy decomposition of a finite-dimensional module I. The crucial role here plays the intersection $R(I)$ of all maximal overmodules of I. Instead of I the modules $R(I)$ and $Syz(I, R(I))$ with smaller differential type or smaller typical differential dimension (see e.g., [11], [12]) are considered in the inductive definition. After that the Loewy decomposition is generalized to infinite-dimensional modules I of differential type $\tau > 0$. It relies on the intersection $R_\tau(I)$ of the classes of maximal overmodules of I with differential type τ, considered up to modules of differential types less than τ. Section 4 deals with parametric-algebraic families of $\mathcal{D}$-modules. They are applied in Section 5 for the discussion of algorithms. In particular the theory outlined in the preceding sections is applied to certain classes of second- and third-order linear pde's with rational function coefficients. An algorithm is presented that accomplishes its Loewy decomposition whenever possible. If it succeeds the solution may be obtained from it.

1. INVARIANCE OF THE SPACE OF SOLUTIONS OF A $\mathcal{D}$-MODULE

Let F be a universal differential field [11] with commuting derivatives $d_1, \dots, d_m$ and $\mathcal{D} = F[d_1, \dots, d_m]$ be the ring of partial differential operators. Denote by $C \subset F$ its subfield of constants. Introduce differential indeterminates $y_1, \dots, y_n$ over F. By Θ denote the commutative monoid generated by $d_1, \dots, d_m$ and by Γ the set of all derivatives θy_i for $\theta \in \Theta, 1 \le i \le n$. We fix also an admissible total ordering $\prec$ on the derivatives [12]. A background in differential algebra may be found in [11, 2, 26, 27].

Let $I \subset \mathcal{D}^n$ be a left $\mathcal{D}$-module. For vectors $g = (g_1, \dots, g_n)$, $v = (v_1, \dots, v_n) \in F^n$ we denote the inner product $gv = (g, v^T) = \sum g_i v_i \in F$. By $V_I = \{v \in F^n : Iv = 0\} \subset F^n$ we denote the space of solutions of I being a C-vector space. A priori V_I depends on the imbedding $I \subset \mathcal{D}^n$. The purpose of this section is to show that actually V_I depends up to an isomorphism just on the factor $\mathcal{D}^n/I$, considered as well up to an isomorphism.

Now let $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$. We say that a $n_1 \times n_2$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathcal{D}$ provides a $\mathcal{D}$-homomorphism from $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$, i.e. $I_1 A \subset I_2$. Clearly one gets a homomorphism of $\mathcal{D}$-modules.

We call $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ to be $\mathcal{D}$-isomorphic if in addition there exists a $n_2 \times n_1$ matrix $B = (b_{ij})$ with $b_{ij} \in \mathcal{D}$ such that $(\mathcal{D}^{n_2}/I_2)B \subset \mathcal{D}^{n_1}/I_1$ and

$$AB|_{(\mathcal{D}^{n_1}/I_1)} = id, \quad BA|_{(\mathcal{D}^{n_2}/I_2)} = id. \qquad (1)$$

For the spaces of solutions $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$ we say that a matrix A provides a *$\mathcal{D}$-homomorphism* if $A(V_{I_2})^T \subset (V_{I_1})^T$ (more precisely, one should talk about a $\mathcal{D}$-homomorphism of the imbeddings $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$). In a similar way, if there exists a $n_2 \times n_1$ matrix B such that $B(V_{I_1})^T \subset (V_{I_2})^T$ and

$$AB|_{V_{I_1}^T} = id, \quad BA|_{V_{I_2}^T} = id \qquad (2)$$

we call V_{I_1}, V_{I_2} to be *$\mathcal{D}$-isomorphic* and denote this by $V_{I_1} \simeq_{\mathcal{D}} V_{I_2}$. The following proposition extends Lemma 2.5 [25] (established for the ordinary case $m = 1$) to finite-dimensional modules.

Proposition 1.1. *i) A matrix A provides a $\mathcal{D}$-homomorphism of $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if and only if it provides $\mathcal{D}$-homomorphisms of V_{I_2} to V_{I_1}.*

ii) $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ are $\mathcal{D}$-isomorphic if and only if V_{I_1} and V_{I_2} are $\mathcal{D}$-isomorphic.

Proof. $i)$ Assume that $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$. We need to verify that $A(V_{I_2})^T \subset (V_{I_1})^T$. The latter is equivalent to the equality $I_1 A (V_{I_2})^T = 0$ which holds because of the inclusion $I_1 A \subset I_2$.

Conversely, assume that $A(V_{I_1})^T \subset (V_{I_1})^T$, then as above $I_1 A (V_{I_2})^T = 0$ which implies $I_1 A \subset I_2$ due to the duality in the differential Zariski topology (see Corollary 1, page 148 in [11], also [26]). Hence $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$.

$ii)$ Assume that (1) holds. One has to verify (2), i. e. for any $v \in V_{I_1}$ to show that $ABv^T = v^T$. The latter holds if and only if for any $g \in \mathcal{D}^{n_1}$ the equality $gABv^T = gv^T$ is true. Equation (1) entails that $gABv^T = (g + g_0)v^T = gv^T$ for a certain vector $g_0 \in I_1$.

We mention that $\mathcal{D}$-isomorphism of $\mathcal{D}$-modules implies isomorphism of the spaces of their solutions in a more general setting, see e.g. [16], [18] (while the converse essentially uses that we deal with a universal differential field).

2. RELATIVE SYZYGIES OF $\mathcal{D}$-MODULES

In Loewy's original decomposition scheme, the largest completely reducible right factors are removed by exact division. In the ring of partial differential operators this is not valid any more. In addition to the relations following from the exact division the syzygies of the right factor have to be taken into account. The proper generalization of the exact quotient is given by the following

Definition 2.1. *(Relative syzygies module) Let $I \subseteq J \subseteq \mathcal{D}^n$ be two $\mathcal{D}$-modules, and let $J =< g_1, \dots, g_t >$. The relative syzygies $\mathcal{D}$-module $Syz(I, J)$ of I and J is $Syz(I, J) = \{(h_1, \dots, h_t) \in \mathcal{D}^t | \sum h_i g_i \in I\}$.*

This definition is more general than the definition of the quotient of $\mathcal{D}$-modules in [13] because we do not require $g_1, \dots, g_t$ to be a Janet basis of J (for a background on Janet basis see e.g. [11, 12, 22]) and in addition it takes into account all relations among $g_1, \dots, g_t$ which put them in I. We notice that in case when $I = 0$ the module $Syz(0, J)$ coincides with the usual syzygies module $Syz(J)$. Our next goal is to show that Definition 2.1 does not depend on the choice of generators $g_1, \dots, g_t$.

Lemma 2.2. *Let $I \subseteq I_1 \subseteq J$ be $\mathcal{D}$-modules. Then $Syz(I_1, J)/Syz(I, J) \simeq I_1/I$.*

Corollary 2.3. *i) $\mathcal{D}^t/Syz(I, J) \simeq J/I$;*
ii) $Syz(I, J)/Syz(J) \simeq I$.

The main goal for introducing the relative syzygies module is the following statement proved in [13] when $g_1, \dots, g_t$ is a Janet basis of J, one can find in [19] another proof.

Lemma 2.4. *With the notation above there holds $V_{Syz(I,J)} \simeq_{\mathcal{D}} V_I/V_J$.*

The following corollary claims that the space of solutions $V_{Syz(I,J)}$ of a relative syzygies module depends just on the factor of $\mathcal{D}$-modules J/I.

Corollary 2.5. *Let $I_1 \subseteq J_1 \subseteq \mathcal{D}^{n_1}$, $I_2 \subseteq J_2 \subseteq \mathcal{D}^{n_2}$. Then $J_1/I_1 \simeq J_2/I_2$ if and only if*

$$V_{Syz(I_1,J_1)} \simeq_{\mathcal{D}} V_{I_1}/V_{J_1} \simeq_{\mathcal{D}} V_{I_2}/V_{J_2} \simeq_{\mathcal{D}} V_{Syz(I_2,J_2)}.$$

Proof. Corollary 2.3 implies that $J_1/I_1 \simeq \mathcal{D}^{q_1}/Syz(I_1, J_1)$ and $J_2/I_2 \simeq \mathcal{D}^{q_2}/Syz(I_2, J_2)$. Both $\mathcal{D}$-isomorphisms $V_{Syz(I_1,J_1)} \simeq_{\mathcal{D}} V_{I_1}/V_{J_1}$ and $V_{Syz(I_2,J_2)} \simeq_{\mathcal{D}} V_{I_2}/V_{J_2}$ follow from Lemma 2.4. Proposition 1.1 entails that $V_{Syz(I_1,J_1)} \simeq_{\mathcal{D}} V_{Syz(I_2,J_2)}$ if and only if $\mathcal{D}^{q_1}/Syz(I_1, J_1) \simeq \mathcal{D}^{q_2}/Syz(I_2, J_2)$ □

Remark 2.6. *Having Janet bases of $I =< f_1, \dots, f_s >$ and of $J =< g_1, \dots, g_t >$ one can construct a Janet basis of $Syz(I, J)$, e. g. cf. Theorem 5.3.7 in [12], also [13]. Briefly to remind, for each f_j there holds $f_j = \sum h_{i,j} g_i$, $1 \le j \le s$ for certain $h_{i,j} \in \mathcal{D}$. Furthermore, for each pair (k, j) with $1 \le k < j \le t$ we represent the Δ-polynomial of g_k and g_j as $lc(g_j)\theta_1 g_k - lc(g_k)\theta_2 g_j = \sum h_{ijk} g_i$ such that the operators $lc(g_j)\theta_1 g_k$ and $lc(g_k)\theta_2 g_j$ have the same leading terms with the minimal possible leading derivative w.r.t. the applied term ordering $\prec$. Then the basis of $Syz(I, J)$ consists of the vectors $(h_{1,j}, \dots, h_{t,j})$, $1 \le j \le s$, and of the vectors*

$$(h_{1jk}, \dots, h_{kjk} - lc(g_j)\theta_1, \dots, h_{jjk} - lc(g_k)\theta_2, \dots, h_{tjk}) \quad (3)$$

for $1 \le k < j \le t$. In the special case $I = 0$, the relative syzygies module $Syz(0, J)$ reduces to the syzygies module of J. Then as in Schreyer's theorem, page 212 of [3], one can show that the constructed basis of $Syz(0, J)$ which consists of vectors of the form (3) constitutes a Janet basis.

We mention also that relying on the algorithm from [8] one can produce a basis of $Syz(I, J)$ starting with arbitrary, not necessarily Janet bases, of I and J, with double-exponential complexity.

Let us denote by H_I the Hilbert-Kolchin polynomial of I w.r.t. the usual filtration by order of derivatives, so $(\mathcal{D}^n)_r = \{f \in \mathcal{D}^n : ord\, f \le r\}$ (cf. page 223 of [12]). The degree $deg(H_I)$ of H_I is called the *differential type* of I [11], page 130 and [12], page 229, and its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I ibid. The differential type denotes the largest number of arguments occuring in any undetermined function of the general solution. The typical differential dimension means the number of functions depending on this maximal number of arguments.

The next theorem can be deduced directly from Theorem 5.2.9 of [12], cf. also Theorem 4.1 in [26].

Theorem 2.7. *Let again $I \subseteq J \subseteq \mathcal{D}^n$. Then $deg\ (H_J) \le deg\ (H_I)$, $deg\ (H_{Syz(I,J)}) \le deg\ (H_I)$ and $deg\ (H_{Syz(I,J)}) = deg\ (H_I - H_J)$, $lc\ (H_{Syz(I,J)}) = lc\ (H_I - H_J)$.*

Definition 2.8. *(Gauge of a module) Let I be a $\mathcal{D}$-module. We call the pair $(deg(H_I), lc(H_I))$ the gauge of I. We say that a module I_1 is of lower gauge than another one I_2 if the pair $(deg(H_{I_1}), lc(H_{I_1}))$ is less than $(deg(H_{I_2}), lc(H_{I_2}))$ in the lexicographic ordering. Taking into account Corollary 2.5 one can talk also about the gauges of the corresponding spaces of solutions V_{I_1} and V_{I_2}.*

The construction of the relative syzygies allows to reduce finding a basis of V_I to finding a basis of V_J and joining it with any solution y of the system $g_i y = w_i$, $1 \le i \le t$ for each element $(w_1, \dots, w_t)$ of a basis of $V_{Syz(I,J)}$. An algorithm for solving the inhomogeneous system $g_i y = w_i$ may be obtained by a proper generalization of Lagrange's variation of constants, see e. g. the textbook [28], page 193-195 if the homogeneous system is known to have a finite-dimensional solution space which will be the case in our applications. Theorem 2.7 implies that both J and $Syz(I, J)$ have gauges not greater than the gauge of I. In the applications in the next section, the gauges of J and $Syz(I, J)$ will be actually lower than the gauge of I. In case of a finite-dimensional ideal I this reduction was exploited in [13].

3. LOEWY DECOMPOSITIONS

Let us first study the case of a finite-dimensional module $I \subset \mathcal{D}^n$ of differential type 0. Consider the intersection $R(I) = J^{(0)} = \cap J$ of all maximal modules $J \supseteq I$. Any intersection of maximal modules will be called a *complete intersection*. $R(I)$ plays a role similar to the role of the radical of two-sided ideals in a ring. Note that there exists a finite number of maximal modules $J_1, \dots, J_q$ for which $J_1 \cap \dots \cap J_q = R(I)$. Indeed, keep taking $J_1, J_2, \dots$ while it is possible to have $dim_C V_{J_1 \cap \dots \cap J_{i+1}} > dim_C V_{J_1 \cap \dots \cap J_i}$ for every $i \ge 1$. Since $dim_C V_I < \infty$ we arrive finally at $J_1, \dots, J_q$ such that $dim_C V_{J_1 \cap \dots \cap J_q \cap J} = dim_C V_{J_1 \cap \dots \cap J_q}$ for any maximal module $J \supseteq I$. Then $J_1 \cap \dots \cap J_q = R(I)$. Applying this procedure to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$, replacing the role of I, which one can compute making use of Remark 2.6, this yields a complete intersection $J^{(1)}$ such that $J^{(1)} = R(I^{(1)}) \supseteq I^{(1)}$. Continuing this way, one obtains successively the complete intersections $J^{(0)}, J^{(1)}, \dots, J^{(s)}$ and the modules $I^{(1)}, \dots, I^{(s)}$ such that $J^{(l)} = R(I^{(l)})$ and $I^{(l+1)} = Syz(I^{(l)}, J^{(l)})$ for

$0 \leq l \leq s-1$, defining $I^{(0)} = I$. In the last step there holds $J^{(s)} = I^{(s)}$. We have $\dim_C V_{I^{(l)}} - \dim_C V_{I^{(l+1)}} = \dim_C V_{J^{(l)}}$ for $0 \leq l \leq s$, defining $I^{(s+1)} = \{0\}$. Thus, $\dim_C V_I = \sum_{0 \leq l \leq s} \dim_C V_{J^{(l)}}$, which provides an upper bound $s < \dim_C V_I$ on the number of steps of the described procedure. The uniquely defined sequences $J^{(0)}, J^{(1)}, \ldots, J^{(s)}$ and $I^{(1)}, \ldots, I^{(s)}$ can be viewed as a *Loewy decomposition* of I. To get the spaces of solutions $V_{J^{(l)}}$, $0 \leq l \leq s$ of the complete intersections $J^{(l)} = \cap_q J_q^{(l)}$ where $J_q^{(l)}$ are maximal modules, we apply proposition 3.1 [26] (see also the beginning of the proof of theorem 4.1 [26], p.483 and [2]) which entails that $V_{J^{(l)}} = \sum_q V_{J_q^{(l)}}$.

Now we proceed to a Loewy decomposition of an infinite-dimensional module $I \subset \mathcal{D}^n$ of differential type $\tau > 0$. To this end, we introduce another concept first.

DEFINITION 3.1. *(Gauge-equivalence) We say that two modules $J_1, J_2 \subset \mathcal{D}^n$ are gauge-equivalent if J_1, J_2 and $J_1 \cap J_2$ are of the same gauge.*

If J_1 and J_2 are gauge-equivalent, then by Theorem 4.1 in [26] also $J_1 + J_2$ is of the same gauge. Gauge equivalence is an equivalence relation. The equivalence class of gauge-equivalent modules of a module J is denoted by $[J]$. If the actual value of the differential type of the elements of a class $[J]$ equals to τ, any two members of it are called *τ-equivalent* (below τ is fixed and $|J|$ means a class of τ-equivalence).

EXAMPLE 3.2. *Let $J_1 =< \partial_x >$, $J_2 =< \partial_{xx}, \partial_{xy} >$ and $J_3 =< \partial_y >$. Then $J_1 \cap J_2 = J_2$, $J_1 + J_2 = J_1$ all of which are of gauge $(1, 1)$. Consequently J_1 and J_2 are gauge-equivalent. The general solution of $z_x = 0$ is $F(y)$, whereas $z_{xx} = z_{xy} = 0$ has general solution $Cx + F(y)$, C being a constant and F an undetermined function of y. Although J_3 is also of gauge (1,1), it is not gauge-equivalent to J_1 because $J_1 \cap J_3 =< \partial_{xy} >$ which is of gauge (1,2).*

We say that $[J_1]$ is *subordinated* to $[J_2]$ if $J_1 \cap J_2$ is τ-equivalent to J_1. One can verify that this relation does not depend on representatives J_1 and J_2 of the classes. We denote this relation by $[J_1] \trianglelefteq [J_2]$. Then $lc(H_{J_1}) \geq lc(H_{J_2})$. If in addition $[J_1] \neq [J_2]$ (we denote this by $[J_1] \triangleleft [J_2]$) then $lc(H_{J_1}) > lc(H_{J_2})$. Hence any increasing chain of τ-equivalence classes stops and one can consider maximal τ-equivalence classes.

For any τ-equivalence classes $[J_1], [J_2]$ satisfying $[J] \trianglelefteq [J_1]$, $[J] \trianglelefteq [J_2]$ one can uniquely define the class $[J_1 \cap J_2]$ such that $[J] \trianglelefteq [J_1 \cap J_2]$. One can verify that $\deg(H_{J_1 \cap J_2}) = \tau$ and the class $[J_1 \cap J_2]$ does not depend on the representatives J_1, J_2.

EXAMPLE 3.3. *Let $J =< \partial_{xyy} >$ with gauge $(1, 3)$, $J_1 =< \partial_x >$ and $J_2 =< \partial_y >$, both with gauge $(1, 1)$. Because $J \cap J_1 = J \cap J_2 = J$ there holds $[J] \trianglelefteq [J_1]$ and $[J] \trianglelefteq [J_2]$. Furthermore $J_1 \cap J_2 =< \partial_{xy} > \equiv J_3$ with gauge $(1, 2)$ and $[J] \trianglelefteq [J_3]$. Because $lc(H_J) = 3$, $lc(H_{J_3}) = 2$ and $lc(H_{J_1}) = lc(H_{J_2}) = 1$, both $[J_1]$ and $[J_2]$ are maximal.*

Now take all τ-maximal classes $[J]$ such that $[I] \trianglelefteq [J]$. Since $J + I$ is τ-equivalent to J (again due to Theorem 4.1 [26]) we can assume without loss of generality that the representatives are chosen in such a way that $I \subseteq J$. We choose consecutively such classes $[J_1], [J_2], \ldots, [J_p]$ while it is possible to have

$$[J_1] \triangleright [J_1 \cap J_2] \triangleright \cdots \triangleright [J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p].$$

Clearly, $p \leq lc(H_I)$. Then for any maximal class $[J]$ for which $[I] \trianglelefteq [J]$, we obtain $[J^{(0)}] \trianglelefteq [J]$. Hence for any finite family $[J_1'], \ldots, [J_q']$ of τ-maximal classes for which $[I] \trianglelefteq [J_l']$, $1 \leq l \leq q$, we conclude that $[J^{(0)}] \trianglelefteq [J_1' \cap \cdots \cap J_q']$. Therefore, the class $[J^{(0)}]$ is defined uniquely and in addition $I \subseteq J^{(0)}$ holds. We say that $J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p$ is *completely τ-reducible*.

We define a Loewy decomposition of I by induction on the gauge of I. As a base of induction when the τ-class $[I]$ is maximal then I provides a Loewy decomposition of itself. When $[I]$ is not maximal one can further apply the described inductive definition of a Loewy decomposition (thereby, replacing the role of I) to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$ (see Section 2) taking into account that either $\deg(H_{I^{(1)}}) < \tau$ or $\deg(H_{I^{(1)}}) = \tau$, and in the latter case $lc(H_{I^{(1)}}) = lc(H_I) - lc(H_{J^{(0)}}) < lc(H_J)$ due to Theorem 2.7; in other words, $I^{(1)}$ is of a lower gauge than I. In case when $\deg(H_{I^{(1)}}) < \tau$ we have $[I] = [J^{(0)}]$ again due to Theorem 2.7 and $[I]$ being completely τ-reducible.

Continuing this way we arrive at a sequence of modules $J^{(0)}, J^{(1)}, \ldots, J^{(q)}$ with non-decreasing differential types such that each module $J^{(l)}$, $0 \leq l \leq q$ is completely $\deg(H_{J^{(l)}})$-reducible. We notice that this sequence is not necessarily unique unlike the Loewy decomposition of a finite-dimensional module. The obtained sequence could be called a *generalized Loewy decomposition* of I. At present we don't possess an algorithm to construct it in general.

4. PARAMETRIC-ALGEBRAIC FAMILIES OF $\mathcal{D}$-MODULES

For the rest of the paper, dealing with the design of algorithms, we assume that the coefficients of the input operators belong to the differential field $F_0 = \overline{\mathbf{Q}}(X_1, \ldots, X_m)$ with derivatives $d_k = \partial/\partial X_k, 1 \leq k \leq m$ and

$$\mathcal{D}_0 = F_0[d_1, \ldots, d_m], \mathcal{D} = F[d_1, \ldots, d_m]$$

where F is a universal extension of F_0. In the sequel we suppose that all the considered algebraic (affine) varieties $W \subset \overline{\mathbf{Q}}^N$ are given in an efficient way, say as in [6]. Namely, $W = \cup W_j$ where W_j are irreducible over $\mathbf{Q}$ components of W, and the algorithms from [6] represent each W_j (of dimension s) in two following ways.

First, we represent W_j by means of a *generic point*, i.e. an isomorphism $\mathbf{Q}(t_1, \ldots, t_s)[\alpha] \simeq \mathbf{Q}(W_j)$ where $\mathbf{Q}(W_j)$ is the field of rational functions on W_j. The elements $t_1, \ldots, t_s \subset \{Z_1, \ldots, Z_N\}$ constitute a basis of transcendency of $\mathbf{Q}(W_j)$ over $\mathbf{Q}$ which can be taken among the coordinates $Z_1, \ldots, Z_N$ of the affine space $\overline{\mathbf{Q}}^N$. The element $\alpha = \sum_{1 \leq l \leq N} \alpha_l Z_l$ for suitable integers α_l is algebraic over the field $\mathbf{Q}(t_1, \ldots, t_s)$ with a minimal polynomial $\phi \in \mathbf{Q}(t_1, \ldots, t_s)[Z]$. The algorithms from [6] yield the ingredients of a generic point explicitly, in other words, $t_1, \ldots, t_s$; $\alpha_1, \ldots, \alpha_N$; ϕ and the rational expressions of Z_l via $t_1, \ldots, t_s, \alpha$, i.e. the rational functions of the form $g_l(t_1, \ldots, t_s, Z)/g(t_1, \ldots, t_s)$, the polynomials $g(t_1, \ldots, t_s), g_l(t_1, \ldots, t_s, Z) \in \mathbf{Q}[t_1, \ldots, t_s, Z]$ being such that $Z_l = g_l(t_1, \ldots, t_s, Z)/g(t_1, \ldots, t_s)$ holds everywhere on W_j.

Second, the algorithms from [6] yield polynomials $h_1, \ldots, h_M \in \mathbf{Q}[Z_1, \ldots, Z_N]$ such that W_j coincides with the variety of all points from $\overline{\mathbf{Q}}^N$ satisfying $h_1 = \cdots = h_M = 0$.

The algorithms from [6] allow to produce the union, intersection, complement of varieties, to get the dimension of W_j, to project a variety (in other words, to eliminate quantifiers), to find all points of W_j if it is finite (zero-dimensional) or to yield any number of points if W_j is infinite (positive-dimensional). Moreover, one extends these algorithms from varieties to constructive sets , i.e. the unions of the sets of the form $W' \setminus W''$ where W', W'' are varieties (in other terms, constructive sets constitute the boolean algebra generated by all the varieties).

Definition 4.1. *(Parametric-algebraic $\mathcal{D}$-modules) We say that a family of $\mathcal{D}$-modules $\mathcal{J} = \{J\} \subset \mathcal{D}^n$ is parametric-algebraic if there is a constructive set $V = \cup V_j \subset \overline{\mathbf{Q}}^N$ for an appropriate N such that $\mathcal{J} = \cup \mathcal{J}_j$ and for any fixed j the following holds. A Janet basis of any $J \in \mathcal{J}_j$ has fixed leading derivatives $lder(J) = lder_j$ and the parametric derivatives $pder(J) = pder_j$, see [13]. Moreover, any element of the Janet basis of J has the form*

$$\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma \tag{4}$$

where $\gamma_0 \in lder_j$ and $A_\gamma \in \mathbf{Q}(Z_1, \dots, Z_N)(X_1, \dots, X_m)$.

When $(Z_1, \dots, Z_N)$ ranges over the constructive set V_j, the set of linear differential operators of the form (4) for all $\gamma_0 \in lder_j$ ranges over the Janet basis for all modules J from $\mathcal{J}_j$. Thus, we have a bijective correspondance between the points of V_j and the modules, or rather their Janet basis) from $\mathcal{J}_j$.

We rephrase in our terms the following proposition which was actually proved in [13].

Proposition 4.2. *([13]). One can design an algorithm which for any finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ finds a parametric-algebraic family of all the factors of I, i.e. the modules $J \subset \mathcal{D}^n$ such that $I \subset J$.*

Lemma 4.3. *One can design an algorithm which for a pair of parametric-algebraic families $\mathcal{I}, \mathcal{J}$ of D-modules yields the parametric-algebraic family of all the pairs (I, J) where $I \in \mathcal{I}, J \in \mathcal{J}$ such that $I \subseteq J$.*

Proof. Let $\{\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma\}_{\gamma_0 \in lder_j}$ be a Janet basis of $\mathcal{J}_j$ and $\{\lambda_0 + \sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda\}_{\lambda_0 \in lder_s}$ be a Janet basis of $\mathcal{I}_s$. Then the condition that $I \subseteq J$ for $I \in \mathcal{I}_s, J \in \mathcal{J}_j$ can be expressed as the existence for each $\lambda_0 \in lder_s$ of operators of the form $\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta \in \mathcal{D}$ where $\theta \prec \theta_0$ and $\lambda_0 = \theta_0 y_i$ for a certain $1 \le i \le n$ such that

$$\lambda_0 + \textstyle\sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda = \sum_{\gamma_0 \in lder_j}(\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta)(\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma) \tag{5}$$

where the external summation in the right-hand side ranges over the elements of the Janet basis of $\mathcal{J}_j$. One can rewrite (5) as a system of linear algebraic equations in the unknowns $C_{\theta,\gamma_0,\lambda_0}$, while the entries of this system are rational functions from $\mathbf{Q}(X_1, \dots, X_m)$ $(Z_1, \dots, Z_N)$. One can find the constructive set $U = U_{j,s} \subset \overline{\mathbf{Q}}^N$ such that for $(Z_1, \dots, Z_N) \in U$ this linear system is solvable. Combining this for all pairs l, s completes the proof. □

Corollary 4.4. *For a finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ one can find a parametric-algebraic family $\mathcal{I}_{max}$ of all maximal $\mathcal{D}$-modules J which contain I.*

Proof. Among the family of all the factors J of I produced in proposition 4.2 one can relying on Lemma 4.3 distinguish all J_0 such that if $J_0 \subseteq J$ then $J_0 = J$ holds. □

5. CONSTRUCTING LOEWY-DECOMPOSITIONS. ALGORITHMS

Now we are able to construct the Loewy decomposition for any finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}_0^n$. According to Corollary 4.4 we determine the intersection $R(I)$ of all maximal modules from $\mathcal{I}_{max}$. To this end we conduct the internal recursion on $\dim_C V_{R(I)}$. Assume that a complete intersection J_0 of several maximal modules from $\mathcal{I}_{max}$ has already been constructed. Applying Lemma 4.3 we test whether there exists a maximal module $J \in \mathcal{I}_{max}$ which does not contain J_0. Then we replace J_0 by the complete intersection $J \cap J_0$ and continue the internal recursion. Finally, we arrive at $R(I)$ and, by external recursion, proceed to the relative syzygies module $Syz(I, R(I))$, provided that the latter is not zero, else halt. Thus, we have shown the following

Corollary 5.1. *For a finite-dimensional D-module $I \subset \mathcal{D}_0^n$ one can construct its Loewy decomposition.*

This construction is the basis in [13] for decomposing finite-dimensional modules. An algorithm has been given there which applies these steps. An implementation may be found in the ALLTYPES system [24].

For general modules the answer is less complete. In [9] proper factorizations and the corresponding decompositions have been considered for second- and third-order operators. Here this approach is extended to the case where genuine factors of such operators do not exist.

Most of the research on finding closed-form solutions of lpde's has been restricted to second-order equations for an unknown function z depending on two arguments x and y. The general linear equation of this kind may be written as

$$Rz_{xx} + Sz_{xy} + Tz_{yy} + Uz_x + Vz_y + Wz = 0 \tag{6}$$

where $R, S, \dots, W$ are from some function differential field which is usually called the base field. Under fairly general constraints for its coefficients it can be shown that it may be transformed either of the following two forms.

$$z_{xy} + A_1 z_x + A_2 z_y + A_3 z = 0, \tag{7}$$

$$z_{xx} + A_1 z_x + A_2 z_y + A_3 z = 0. \tag{8}$$

In this section it is always assumed that all $A_k \in \mathbf{Q}(x, y)$. Any solution scheme is closely related to the question what type of solutions are searched for. For linear ode's the answer is well known. The general solution is a linear combination of a fundamental system over the constants. For pde's the answer is much more involved. Equations of the form (7) may allow solutions of either of the two forms

$$f_0(x, y)F(x) + f_1(x, y)F'(x) + \ldots + f_m(x, y)F^{(m)}(x), \tag{9}$$

$$g_0(x, y)G(y) + g_1(x, y)G'(y) + \ldots + g_n(x, y)G^{(n)}(y) \tag{10}$$

where the f_k, g_k are determined by the given equation, and $F(x)$ and $G(y)$ are undetermined functions of the respective argument. The existence of either type of solution, or of both types, depends on the values of the coefficients A_k. To decide their existence is already highly nontrivial. Moreover there may be solutions with integrals involving the undetermined elements. An algorithm is described now which performes these steps for certain pde's of second or third order. Equation (7) is written as $D_{xy}z = 0$ where

$$D_{xy} \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3. \tag{11}$$

This case has been studied most thorougly in the literature. It will be discussed first. The principal ideal $< D_{xy} >$ is of gauge $(1, 2)$. There may exist operators forming a Janet base in combination with (11) which are of the form

$$D_{x^m} \equiv \partial_{x^m} + a_1\partial_{x^{m-1}} + \ldots + a_{m-1}\partial_x + a_m \qquad (12)$$

$$\text{or} \quad D_{y^n} \equiv \partial_{y^n} + b_1\partial_{y^{n-1}} + \ldots + b_{n-1}\partial_y + b_n \qquad (13)$$

with m and n positive integers. Usually it is a difficult problem to construct new operators which extend a set of given ones to form the Janet base of a larger ideal. However, due to the special structure of the problem, the auxiliary systems for the unknown coefficients a_j and b_j in (12) and (13) may always be solved as is shown next.

Proposition 5.2. *Let an operator of the form (11) be given. The following types of overideals may be constructed.*

a) *If $n \geq 2$ is a natural number, it may be decided whether there exists an operator (13) such that (11) and (13) combined form a Janet base. If the answer is affirmative, the operator (13) may be constructed explicitly with coefficients $b_i \in \mathbf{Q}(x, y)$, the ideal $< D_{xy}, D_{y^n} >$ is of gauge (1,1).*

b) *If $m \geq 2$ is a natural number, it may be decided whether there exists an operator (12) such that (11) and (12) combined form a Janet base. If the answer is affirmative, the operator (12) may be constructed explicitly with coefficients $a_i \in \mathbf{Q}(x, y)$, the ideal $< D_{xy}, D_{x^m} >$ is of gauge (1,1).*

Proof. The proof will be given for case a). If the operator (11) is derived repeatedly wrt. y, and the reductum is reduced in each step wrt. (11), $n - 2$ equations of the form

$$\partial_{xy^k} + R_k\partial_x + P_{k,k}\partial_{y^k} + P_{k,k-1}\partial_{y^{k-1}} + \ldots + P_{k,0} \qquad (14)$$

for $2 \leq k \leq n - 1$ may be obtained. All coefficients R_k and $P_{i,j}$ are differential polynomials in the ring $\mathbf{Q}\{A_1, A_2, A_3\}$. There is no reduction wrt. (13) possible. Deriving the last expression once more wrt. y and reducing the reductum wrt. both (7) and (13) yields

$$\begin{aligned}\partial_{xy^n} + R_n\partial_x + (P_{n,n-1} - P_{n,n}b_1)\partial_{y^{n-1}}& \\ +(P_{n,n-2} - P_{n,n}b_2)\partial_{y^{n-2}} + ...& \qquad (15)\\ +(P_{n,1} - P_{n,n}b_{n-1})\partial_y + P_{n,0} - P_{n,n}b_n.&\end{aligned}$$

In the first derivative of (13) wrt. x

$$\begin{aligned}\partial_{xy^n} + b_{1,x}\partial_{y^{n-1}} + b_{2,x}\partial_{y^{n-2}} + \ldots + b_{n-1,x}\partial_y + b_{n,x}& \\ +b_1\partial_{xy^{n-1}} + b_2\partial_{xy^{n-2}} + \ldots + b_{n-1}\partial_{xy} + b_n\partial_x&\end{aligned}$$

the terms containing derivatives of the form ∂_{xy^k} for $1 \leq k \leq n - 1$ may be reduced wrt. (14) or (7) with the result

$$\begin{aligned}&\partial_{xy^n} + (b_{1,x} - P_{n-1,n-1}b_1)\partial_{y^{n-1}}\\ &+(b_{2,x} - P_{n-1,n-2}b_1 - P_{n-2,n-2}b_2)\partial_{y^{n-2}}\\ &\qquad\vdots \qquad\qquad\qquad \vdots\\ &+(b_{n-1,x} - P_{n-1,1}b_1 - P_{n-2,1}b_2 \ldots - P_{2,1}b_{n-2} - A_2b_{n-1})\partial_y\\ &+b_{n,x} - P_{n-1,0}b_1 - P_{n-2,0}b_2 - \ldots - P_{2,0}b_{n-2} - A_3b_{n-1}\\ &+(b_n - R_{n-1}b_1 - R_{n-2}b_2 - \ldots - R_2b_{n-2} - A_1b_{n-1})\partial_x.\end{aligned}$$

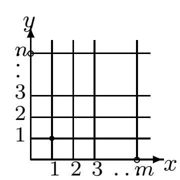
If this expression is subtracted from (15), the coefficients of the derivatives must vanish in order that (7) and (13) form a Janet base. The resulting system of equations is

$$\begin{aligned}b_{1,x} + (P_{n,n} - P_{n-1,n-1})b_1 - P_{n,n-1} = 0,\\ b_{2,x} - P_{n-1,n-2}b_1 + (P_{n,n} - P_{n-2,n-2})b_2 - P_{n,n-2} = 0,\\ \vdots \qquad\qquad\qquad\qquad \vdots\\ b_{n-1,x} - P_{n-1,1}b_1 - \ldots + (P_{n,n} - A_2)b_{n-1} - P_{n,1} = 0,\\ b_{n,x} - P_{n-1,0}b_1 - \ldots - A_3b_{n-1} + P_{n,n}b_n - P_{n,0} = 0,\\ b_n - R_{n-1}b_1 - R_{n-2}b_2 - \ldots - R_2b_{n-2} - A_1b_{n-1} = 0.\end{aligned}$$

The last equation may be solved for b_n. Substituting it into the equation with leading term $b_{n,x}$, and eliminating the first derivatives $b_{j,x}$ for $j = 1, \ldots, n-1$ by means of the preceding equations, it may be solved for b_{n-1}. Proceeding in this way, due to the triangular structure, finally b_1 is obtained from the equation with leading term $b_{2,x}$. Backsubstituting these results, all b_k are explicitly known. Substituting them into the first equation, a constraint for the coefficients A_i expressing the condition for the existence of a Janet base comprising (7) and (13) is obtained. The proof for case b) is similar and is therefore omitted. □

Goursat [5], Section 110, describes a method for constructing a linear ode which is in involution with a given second order equation $z_{xy} + az_x + bz_y + cz = 0$. The advantage of the method given above is that it may be applied to many other problems, e. g. exactly the same strategy works for the third-order equations discussed below. It is not obvious how to generalize Goursat's scheme to any other case beyond the second-order equation considered by him.

Case a), $n = 1$ and case b), $m = 1$, have been discussed in detail in [9]. The corresponding ideals are maximal and principal, because they are generated by $\partial_y + a_1$ and $\partial_x + b_1$ respectively. The term *factorization* is applied in these cases in the proper sense because the obvious analogy to ordinary differential operators where all ideals are principal. For any value $m > 1$ or $n > 1$ the overideals are $J_m =< D_{xy}, D_{x^m} >$ or $J_n =< D_{xy}, D_{y^n} >$. For any fixed values $m_1 < m_2$, the corresponding ideals obey $J_{m_2} \subset J_{m_1}$, and similary for values of n. This situation becomes particularly clear from the following graph.



The heavy dot at $(1, 1)$ represents the leading derivative ∂_{xy} of the given equation. If a second equation with leading derivative ∂_{x^m} represented by the circle at $(m, 0)$ exists, the ideal is enlarged by the corresponding operator. For $m = 1$ this ideal contains the original operator with leading derivative ∂_{xy}, i. e. this operator is redundant. This shows clearly how the conventional factorization corresponding to a first-order operator is obtained as a special case for any m. A similar discussion applies to additional equations with leading derivative ∂_{x^n}.

Next the algebraic approach will be applied third to order equations of the form $D_{xyy}z = 0$ where

$$D_{xyy} \equiv \partial_{xyy} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5. \quad (16)$$

The ideal $< D_{xyy} >$ is of gauge $(1, 3)$. Proper right factors of differential type 1 and of first or second order may be obtained by Corollary 4.3 of [9]. For completeness they are given next without proof.

PROPOSITION 5.3. *An operator of the form (16) generates an ideal $< D_{xyy} >$ of gauge $(1,3)$. It may have the following proper right factors of order two or one.*

a) If $2A_{2,y}+A_1A_2-A_4\neq 0$, $b_{1,y}-b_1^2+A_1b_1-A_3=0$,

$$b_1=\frac{1}{2A_{2,y}+A_1A_2-A_4}(A_{2,yy}+2A_{2,y}A_1 +A_2A_{1,y}-A_{4,y}-A_1A_4-A_2A_3+A_1^2A_2)$$

a right factor $\partial_{xy}+b_1\partial_x+b_2\partial_y+b_3$ exists, $b_2=A_2$, $b_3=A_2b_1+A_4-A_{2,y}-A_1A_2$.

b) If $2A_{2,y}+A_1A_2-A_4=0$ and $A_5-A_{2,yy}-A_{2,y}A_1-A_2A_3=0$, a right factor $\partial_{xy}+b_1\partial_x+b_2\partial_y+b_3$ exists where b_1 is a solution of $b_{1,y}-b_1^2+A_1b_1-A_3=0$, and $b_2=A_2$, $b_3=A_2b_1+A_{2,y}$.

c) If $A_4-2A_{2,y}-A_1A_2=0$ and $A_5-A_{2,yy}-A_{2,y}A_1-A_2A_3=0$, a right factor ∂_x+b exists with $b=A_2$.

d) If $A_4-A_1A_2-A_{1,x}\neq 0$, $b_y-b^2+A_1b-A_3=0$ where $b=\dfrac{A_5-A_2A_3-A_{3,x}}{A_4-A_1A_2-A_{1,x}}$, a right factor ∂_y+b exists.

e) If $A_4-A_1A_2-A_{1,x}=0$ and $A_5-A_2A_3-A_{3,x}=0$, a right factor ∂_y+b exists where b is a solution of $b_{1,y}-b_1^2+A_1b-A_3=0$.

The ideals generated in case a) and b) are of gauge (1,2), in the remaining cases they are of gauge (1,1).

If such a factor does not exist, over-ideals of the form $< D_{xyy}, D_{x^m} >$ or $< D_{xyy}, D_{y^n} >$ may be searched for.

PROPOSITION 5.4. *Let an operator of the form (16) be given. The following types of overideals of differential type 1 may be constructed with coefficients $a_i, b_i \in \mathbf{Q}(x,y)$.*

a) If $n\geq 2$ is a natural number, it may be decided whether there exists an operator (13) such that (16) and (13) combined form a Janet base. If the answer is affirmative, the operator (13) may be constructed explicitly.

b) If $m\geq 2$ is a natural number, it may be decided whether there exists an operator (12) such that (16) and (12) combined form a Janet base. If the answer is affirmative, the operator (12) may be constructed explicitly.

The results obtained up to now are combined to produce the algorithm *DecomposeLpde* which returns the most complete decomposition for any operator of the form (7) or (16).

Algorithm *DecomposeLpde*(L,d). Given an operator L of the form (7) or (16) generating $I =< L >$, its decomposition into overideals of differential type 1 and with leading derivative of order not higher than d is returned.

$S1$: *Proper factorization.* Determine right factors $f_1, f_2, \ldots$ of L as described in Corollary 3.3. If any are found, collect them as $F := \{f_1, f_2, \ldots\}$.

$S2$: *Extend ideal.* If step $S1$ failed, apply Proposition 5.2 or 5.4 in order to construct operators $g_1, g_2, \ldots$ of the form (12) or (13) with $m\leq d$ and $n\leq d$, beginning with $m=n=2$ and increasing its value stepwise by 1 until d is reached. If any are found, assign them to $G := \{g_1, g_2, \ldots\}$. If F and G are empty return L.

$S3$: *Completely reducible?* If $J := Lclm(F) =< L >$ return F, else if for the elements of G there holds $J := Lclm(< L, g_1 >, < L, g_2 >, \ldots) =< L >$, return G.

$S4$: *Relative syzygies.* Determine generators of $S := Syz(I,J)$ and transform it into a Janet base. If F is not empty return (S,F) else return (S,G).

This algorithm has been implemented in ALLTYPES which may be accessed over website `www.alltypes.de` [24]. From this decomposition large classes of solutions of an equation $Lz=0$ may be obtained. In the completely reducible case, from the operators returned in step $S3$ solutions may be constructed as described in [9]. If L is not completely reducible, the result of step $S4$ is applied as follows. From F or G a partial solution is obtained similar as in the previous case. Solving the equations corresponding to S and taking the result as inhomogeneity for F or G respectively yields an additional part of the solution. This proceeding may fail if not all of the equations which occur can be solved. In these cases only a partial solution is obtained. The following examples have been treated according to this proceeding. The first one which is due to Forsyth. It shows how complete reducibility has its straightforward generalization if there are no proper factors.

EXAMPLE 5.5. *(Forsyth 1906) Define*

$$D_{xy}\equiv \partial_{xy}+\frac{2}{x-y}\partial_x-\frac{2}{x-y}\partial_y-\frac{4}{(x-y)^2}$$

which generates the principal ideal $I =< D_{xy} >$ of gauge (1,2). The equation $D_{xy}z=0$ has been considered in [4], vol. VI, page 80. In step $S1$ no first-order factor is obtained. Step $S2$ shows that there exist both generators

$$D_{xx}\equiv \partial_{xx}-\frac{2}{x-y}\partial_x+\frac{2}{(x-y)^2},$$
$$D_{yy}\equiv \partial_{yy}+\frac{2}{x-y}\partial_y+\frac{2}{(x-y)^2}$$

such that the ideals $J_1 =< D_{xy}, D_{xx} >$ and $J_2 =< D_{xy}, D_{yy} >$, each of gauge (1,1), are generated by a Janet base. In step $S3$ it is found that $I = Lclm(J_1, J_2)$, i.e. I is completely reducible, the sum ideal is $J_1+J_2 =< D_{xy}, D_{xx}, D_{yy} >$. The general solution of $D_{xx}z=0$ is $C_1(x-y)+C_2x(x-y)$, $C_{1,2}$ are undetermined functions of y. Substitution into $D_{xy}z=0$ yields $C_{1,y}+yC_{2,y}-C_2=0$. They may be represented as $C_1=2F(y)-yF'(y)$ and $C_2=F'(y)$. Consequently the solution $z_1=2(x-y)F(y)+(x-y)^2F'(y)$ is obtained. The equation $D_{yy}z=0$ has general solution $C_1(y-x)+C_2y(y-x)$, $C_{1,2}$ are undetermined functions of x now. Similar as above, the solution $z_2=2(y-x)G(x)+(y-x)^2G'(x)$ is obtained. The general solution of $D_{xy}z=0$ is z_1+z_2.

The following example by Imschenetzky has been reproduced in many places in the literature.

EXAMPLE 5.6. *(Imschenetzky 1872) The equation $(\partial_{xy}+xy\partial_x-2y)z=0$ has been considered in [10]. Step $S1$ shows again that there are no first-order right factors. According to step $S2$, an operator of the form (13) with $n\leq 3$ does not exist. However, for $m=3$ there is an operator ∂_{xxx} such that the ideal $< \partial_{xy}+xy\partial_x-2y, \partial_{xxx} >$ of gauge (1,1) is generated by a Janet base. The equation $z_{xxx}=0$ has the general solution $C_1+C_2x+C_3x^2$ where the C_i, $i=1,2,3$ are constants wrt. x. Substituting it into the above equation and equating the coefficients of x to zero leads to the system $C_{2,y}-2yC_1=0$, $C_{3,y}-\frac{1}{2}yC_2=0$. The C_i may be represented as $C_1=\frac{1}{y^2}F''-\frac{1}{y^3}F'$, $C_2=\frac{2}{y}F'$, $C_3=F$, F is an undetermined function of y, $F'\equiv dF/dy$. It yields the solution $z_1=x^2F(y)+\dfrac{2xy^2-1}{y^3}F'(y)+\dfrac{1}{y^2}F''(y)$ of the given equation. In step $S4$, from the ideals $I =<$*

$\partial_{xy} + xy\partial_x - 2y >$ *and* $J =< \partial_{xy} + xy\partial_x - 2y, \partial_{xxx} >$ *the relative syzygy module* $Syz(I, J) =< (1, 0), (\partial_{xx}, -\partial_y - xy) >=< (1, 0), (0, \partial_y + xy) >$ *of gauge (1,1) is constructed. Its solution* $(0, G(x)s(x, y))$ *with* $s(x, y) = \exp(-\frac{1}{2}xy^2)$ *and* $G(x)$ *an undetermined function of* x *yields the solution*

$$z_2 = \frac{1}{2}\int G(x)s(x,y)x^2dx \\ -x\int G(x)s(x,y)xdx + \frac{1}{2}x^2\int G(x)s(x,y)dx$$

of the original equation, its general solution is $z_1 + z_2$.

The last example is a third-order equation which allows a single over-ideal generated by ∂_{xxx}.

Example 5.7. *Let the third-order operator*

$$D_{xyy} \equiv \partial_{xyy} + (x+y)\partial_{xy} + (x+y)\partial_x - 2\partial_y - 2$$

be given. It generates the principal ideal $I =< D_{xyy} >$ *of gauge (1,3). Step* $S1$ *does not yield any right factors of order one or two. In step* $S2$ *an operator of the form (13) and* $n \leq 5$*, or an operator of the form (12) for* $m \leq 2$ *is not found. However, for* $m = 3$ *there is an operator* $D_{xxx} \equiv \partial_{xxx}$ *such that the ideal* $J =< D_{xyy}, D_{xxx} >$ *of gauge (1,1) is generated by a Janet base. The equations* $D_{xyy}z = 0$ *and* $D_{xxx}z = 0$ *yield the solution*

$$z_1 = [(x+y)^2 - 2(x+y) + 2]F(y) + 2(x+y-1)F'(y) + F''(y)$$

where F *is an undetermined function of* y*. In step* $S4$*,* I *and* J *yield the relative syzygy module of gauge (1,2)*

$$Syz(I, J) =< (1, 0), (\partial_{xx}, -\partial_{yy} - (x+y)\partial_y - x - y) > \\ =< (1, 0), (0, \partial_{yy} + (x+y)\partial_y + x + y) > .$$

Its solution is $G(x)s(x,y) + H(x)s(x,y)\int e^{-y}\frac{dy}{s(x,y)}$*, where* $s(x, y) = \exp(-\frac{1}{2}(x+y-2)^2 - y)$ *and* G, H *are undetermined functions of* x*. According to the discussion in the Introduction one finally obtains*

$$z_2 = \frac{1}{2}\int G(x)s(x,y)x^2dx \\ -x\int G(x)s(x,y)xdx + \frac{1}{2}x^2\int G(x)s(x,y)dx$$

and for z_3 *an identical expression with* $G(x)$ *replaced by* $H(x)$ *and* $s(x, y)$ *by* $s(x,y)\int e^{-y}\frac{dy}{s(x,y)}$*. The general solution of the given equation* $D_{xyy}z = 0$ *is* $z_1 + z_2 + z_3$.

6. CONCLUSION

The results presented in this article allow decomposing partial differential operators of the form (7) or (16) into components of lower gauge. If such a decomposition is found, it may be applied to determine the general solution of the corresponding pde, or at least some parts of it.

It is highly desirable to develop a similar scheme to large classes of modules of partial differential operators. The possible types of overmodules can always be determined. The hard part is to identify those for which generators may be constructed algorithmically. An important field of application is the symmetry analysis of nonlinear pde's, because the determining equations of these symmetries are linear homogeneous pde's [22]. Another problem is to find an upper bound for the order d in algorithm *DecomposeLpde*. It would mean that full classes of over-modules could be excluded. On the other hand, a negative answer would be an evidence that this problem could be undecidable

7. REFERENCES

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