Loewy Decomposition of Third-Order Linear PDE's in the Plane

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ABSTRACT

Loewy's decomposition of a linear ordinary differential operator as the product of largest completely reducible components is generalized to partial differential operators of order three in two variables. This is made possible by considering the problem in the ring of partial differential operators where both left intersections and right divisors of left ideals are not necessarily principal. Listings of possible decomposition types are given. Many of them are illustraded by worked out examples. Algorithmic questions and questions of uniqueness are discussed in the Summary.

Categories and Subject Descriptors

I.1 [**Symbolic and Algebraic Manipulation**]: Applications

General Terms

Algorithms

Keywords

D-module, Loewy decomposition, Janet basis

1. INTRODUCTION

About one hundred years ago Loewy proved the fundamental result that any ordinary differential operator may be represented uniquely as the product of largest completely reducible operators, i.e. operators that are the left intersection of irreducible operators of lower order [10]; see also Chapter 2 of the book [12]. This decomposition provides a detailed understanding of the structure of the solution space of the corresponding differential equation. Therefore it would be highly desirable to generalize it to partial differential operators as well. Amazingly this has never really been done since Loewy's original work. In this article Loewy decompositions of third-order operators in two variables are considered in detail; the possible limitations are also discussed.

ISSAC'08, July 20–23, 2008, Hagenberg, Austria.

In the subsequent section some basic notations from differential algebra are introduced; details may be found in the book by Kolchin [9] or the articles by Buium and Cassidy [2] or Quadrat [11]. The main part of the article is organized according to leading derivatives of the respective operators.

2. BASIC DIFFERENTIAL ALGEBRA

In this section some basic terminology from differential algebra is introduced. Rings of differential operators $\mathcal{D} \equiv \mathcal{F}[\partial_x, \partial_y]$ or $\mathcal{D} \equiv \mathbb{Q}(x,y)[\partial_x, \partial_y]$ are considered; $\mathcal{F}$ is a universal differential field. $\mathcal{F}$ or $\mathbb{Q}(x,y)$ are called the *base field.* Let the left ideal I, or simply ideal I, be generated by elements $l_i \in \mathcal{D}$, $i = 1, \ldots, p$. Then one writes $I = \langle l_1, \ldots, l_p \rangle$. As a rule, the l_i are assumed to form a Janet basis in the term order *grlex* with $x > y$. If z is some differential indeterminate, $l_i z = 0$, $i = 1, \ldots, p$, is the corresponding system of pde's. If only the leading terms of the generators of an ideal are of interest the non-leading terms are omitted; it is denoted by $\langle ... \rangle_{LT}$. Let $I \subseteq \mathcal{F}[\partial_x, \partial_y]$ be an ideal and H_I its Hilbert-Kolchin polynomial ([9], page 130; [2], page 602). The degree $deg(H_I)$ of H_I is called the *differential type* of I. Its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I. The pair $(deg(H_I), lc(H_I))$ has been baptized the *gauge* of the ideal I [7].

Let I and J be two ideals. Important constructions are the greatest common right divisor $Gcrd(I,J)$ or the sum ideal; and the least common left multiple $Lclm(I,J)$ or the left intersection. In [7] it has been shown how they are computed algorithmically. Finally the *relative syzygy module* $Syz(I,J)$ of I and $J = \langle g_1, \ldots, g_q \rangle$ is generated by

$$\{h \equiv (h_1, \ldots, h_q) \in \mathcal{D}^q | h_1 g_1 + \ldots + h_q g_q \in I\}.$$

Define two ordinary differential operators by

$$D_{x^m} \equiv d_{x^m} + a_1 d_{x^{m-1}} + \ldots + a_{m-1} d_x + a_m,$$
$$D_{y^n} \equiv d_{y^n} + b_1 d_{y^{n-1}} + \ldots + b_{n-1} d_y + b_n;$$

m and n are natural numbers not less than 2. Several ideals generated by an operator of order three and one of these operators will occur later in this article. A special notation is introduced for them as shown in the table below. These ideals have an important meaning as divisors in the decompositions to be discussed in later chapters. Due to its close relation to the iteration scheme introduced by Laplace, see [6], vol. II, Chapter V, it is suggested to call them *Laplace divisors*. The pair of upper indices of the ideals in Table 1 denotes the gauge of the respective ideal.