Absolute Factoring of Non-holonomic Ideals in the Plane

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ABSTRACT

We study *non-holonomic* overideals of a left differential ideal $J \subset F[\partial_x, \partial_y]$ in two variables where F is a differentially closed field of characteristic zero. One can treat the problem of finding non-holonomic overideals as a generalization of the problem of factoring a linear partial differential operator. The main result states that a principal ideal $J = \langle P \rangle$ generated by an operator P with a separable symbol $symb(P)$ has a finite number of maximal non-holonomic overideals; the symbol is an algebraic polynomial in two variables. This statement is extended to non-holonomic ideals J with a separable symbol. As an application we show that in case of a second-order operator P the ideal $\langle P \rangle$ has an infinite number of maximal non-holonomic overideals iff P is essentially ordinary. In case of a third-order operator P we give sufficient conditions on $\langle P \rangle$ in order to have a finite number of maximal non-holonomic overideals. In the Appendix we study the problem of finding non-holonomic overideals of a principal ideal generated by a second order operator, the latter being equivalent to the Laplace problem. The possible application of some of these results for concrete factorization problems is pointed out.

AMS Subject Classifications: 35A25, 35C05, 35G05

Keywords: differential non-holonomic overideals, Newton polygon, formal series solutions.

1. FINITENESS OF THE NUMBER OF MAXIMAL NON-HOLONOMIC OVER-IDEALS OF AN IDEAL WITH SEPARABLE SYMBOL

Let F be a differentially closed field (or universal differential field in terms of [8], [9]) with derivatives ∂_x and ∂_y; let $P = \sum_{i,j} p_{i,j} \partial_x^i \partial_y^j \in F[\partial_x, \partial_y]$ be a partial differential operator of order n. Considering e.g. the field of rational functions $\mathbb{Q}(x, y)$ as F is a quite different issue. The *symbol* is defined by $symb(P) = \sum_{i+j=n} p_{i,j} v^i w^j$; it is a homogeneous algebraic polynomial of degree n in two variables. The degree of its Hilbert-Kolchin polynomial $ez + e_0$ is called its *differential type*; its leading coefficient is called the *typical differential dimension* [8]. A left ideal $I \subset F[\partial_x, \partial_y]$ is called *non-holonomic* if its differential type equals 1. We study maximal non-holonomic overideals of a principal ideal $\langle P \rangle \subset F[\partial_x, \partial_y]$. Obviously there is an infinite number of maximal *holonomic* overideals of $\langle P \rangle$: for any solution $u \in F$ of $Pu = 0$ we get a holonomic overideal $\langle \partial_x - u_x/u, \partial_y - u_y/u \rangle \supset \langle P \rangle$. We assume w.l.o.g. that $symb(P)$ is not divisible by ∂_y; otherwise one can make a suitable transformation of the type $\partial_x \to \partial_x,\ \partial_y \to \partial_y + b\partial_x,\ b \in F$. In fact choosing b from the subfield of constants of F is possible.

Clearly, factoring an operator P can be viewed as finding principal overideals of $\langle P \rangle$; we refer to factoring over a universal field F as *absolute factoring*. Overideals of an ideal in connection with Loewy and primary decompositions were considered in [6].

Following [4] consider a homogeneous polynomial ideal $symb(I) \subset F[v, w]$ and attach a homogeneous polynomial $g = GCD(symb(I))$ to I. Lemma 4.1 [4] states that $deg(g) = e$. As above one can assume w.l.o.g. that w does not divide g.

We recall that the Ore ring $R = (F[\partial_y])^{-1}\ F[\partial_x, \partial_y]$ (see [1]) consists of fractions of the form $\beta^{-1} r$ where $\beta \in F[\partial_y]$, $r \in F[\partial_x, \partial_y]$, see [3], [4]. We also recall that one can represent $R = F[\partial_x, \partial_y]\ (F[\partial_y])^{-1}$, and two fractions are equal, $\beta^{-1} r = r_1 \beta_1^{-1}$, iff $\beta r_1 = r \beta_1$ [3], [4].

For a non-holonomic ideal I denote ideal $\overline{I} = RI \subset R$. Since the ring R is left-euclidean (as well as right-euclidean) with respect to ∂_x over the skew-field $(F[\partial_y])^{-1}\ F[\partial_y]$, we conclude that the ideal $\overline{I}$ is principal. Let $\overline{I} = \langle r \rangle$ for suitable $r \in F[\partial_x, \partial_y] \subset R$ (cf. [4]). Lemma 4.3 [4] implies that $symb(r) = w^m g$ for a certain integer $m \geq 0$ where g is not divisible by w.

Now we expose a construction introduced in [4]. For a family of elements $f_1, \dots, f_k \in F$ and rational numbers $s_i \in \mathbb{Q}$, $1 > s_2 > \dots > s_k > 0$ we consider a D-module being a vector space over F with a basis $\{G^{(s)}\}_{s \in \mathbb{Q}}$ where the derivatives of

$$G^{(s)} = G^{(s)}(f_1, \dots, f_k;\ s_2, \dots, s_k)$$

ISSAC'10, July 20–23, 2010, München.