Factoring Zero-dimensional Ideals of Linear Partial Differential Operators

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ABSTRACT

We present an algorithm for factoring a zero-dimensional left ideal in the ring $\bar{\mathbb{Q}}(x,y)[\partial_x, \partial_y]$, i.e. factoring a linear homogeneous partial differential system whose coefficients belong to $\bar{\mathbb{Q}}(x,y)$, and whose solution space is finite-dimensional over $\bar{\mathbb{Q}}$. The algorithm computes all the zero-dimensional left ideals containing the given ideal. It generalizes the Beke-Schlesinger algorithm for factoring linear ordinary differential operators, and uses an algorithm for finding hyperexponential solutions of such ideals.

1. INTRODUCTION

For various reasons *linear* differential equations have been of particular importance in the history of mathematics. First of all, the problems connected with them are much easier than those for nonlinear equations. Second, many nonlinear problems may be linearized in some way such that the results of the former theory may be applied to them. This is especially true for Lie's symmetry analysis of differential equations which reduces the problem of solving nonlinear ordinary differential equations (ode's) with a sufficiently large number of symmetries to the study of certain systems of linear partial differential equations (pde's). The problem of finding conservation laws for nonlinear pde's also leads to systems of linear pde's.

It has been possible to generalize many concepts from commutative algebra suitably such that they may be applied to linear ode's, e.g. the greatest common divisor and least common multiple, the concept of reducibility and factorization which finally led to the theory of Picard and Vessiot and differential Galois theory. This is true to a much less extent for systems of linear pde's. In order to obtain manageable problems, we have to specialize them further. The constraint that the general solution depends on a finite number of constants, i.e. that it may be represented as a linear combination with constant coefficients of a finite number of special solutions which form a fundamental system, turns out to be appropriate. It allows us to generalize many concepts from the theory of linear ode's in an almost straightforward manner to these pde's. Furthermore, they arise from different research areas such as: symmetry analysis, holonomic differential systems [9] and representations of functions in terms of pde's and initial values [3].

It is the purpose of this paper to describe a generalization of the Beke-Schlesinger factorization algorithm [1, 10] to systems of linear homogeneous pde's in one dependent and two independent variables with a finite-dimensional solution space. The base field consists of the rational functions in the independent variables with algebraic number coefficients. Such an algorithm may have interesting applications in the above-mentioned areas. In principle, most problems related to such systems of pde's reduce, as shown by S. Lie, to corresponding problems for linear ode's. However, such "reduction" may be nontrivial and usually leads to solving or factoring linear ode's with parameters. This makes many known algorithms fail. The algorithm presented in this paper avoids such complications.

In this paper the following notation will be used: $\bar{\mathbb{Q}}$ stands for the algebraic closure of the field of rational numbers, $\mathbf{K}$ for the differential field $\bar{\mathbb{Q}}(x,y)$ with usual derivation operators ∂_x and ∂_y, and $\mathbf{K}[\partial_x, \partial_y]$ for the ring of linear partial differential operators generated by ∂_x and ∂_y over $\mathbf{K}$. By an ideal in $\mathbf{K}[\partial_x, \partial_y]$ we mean a left ideal in $\mathbf{K}[\partial_x, \partial_y]$. A system of linear homogeneous pde's in ∂_x and ∂_y over $\mathbf{K}$ can be naturally identified with an ideal in $\mathbf{K}[\partial_x, \partial_y]$ (see, e.g. [9, §1.1]). A solution of a system is understood as an element of a universal differential field extension $\mathbf{E}$ of $\mathbf{K}$, which is annihilated by the operators in the corresponding ideal of $\mathbf{K}[\partial_x, \partial_y]$. A system has a finite-dimensional solution space over $\bar{\mathbb{Q}}$ if and only if the corresponding ideal is zero-dimensional (abbreviated as: 0-dim). This point of

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