A Factorization Algorithm for Linear Ordinary Differential Equations

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Abstract. The reducibility and factorization of linear homogeneous differential equations are of great theoretical and practical importance in mathematics. Although it has been known for a long time that factorization is in principle a decision procedure, its use in an automatic differential equation solver requires a more detailed analysis of the various steps involved. Especially important are certain auxiliary equations, the so-called *associated equations.* An upper bound for the degree of its coefficients is derived. Another important ingredient is the computation of optimal estimates for the size of polynomial and rational solutions of certain differential equations with rational coefficients. Applying these results, the design of the e .factorization algorithm *LODEF* and its implementation in the Scratchpad II Computer Algebra System is described.

1 Reducibility of Differential Equations

The concept of *reducibility of a differential equation* came into existence in the second half of the last century when many efforts were made to obtain a theory for solving differential equations by analogy to that for algebraic equations which had been created by Lagrange and Galois. According to Frobenius, a linear homogeneous differential equation $P(y)$ with coefficients from a given field is said to be *reducible* if there exists another linear equation $Q(y)$ of lower order with coefficients of the same type which has its solutions in common with $P(y)$. If an equation is not reducible it is called *irreducible.* A reducible equation may be decomposed according to

From: Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation, ISSAC'89, Portland, Oregon, ACM Press, 1989, Gaston Gonnet, ed.

$$P(y) = R(Q(y)), \qquad or \qquad P = RQ \tag{1}$$

for short where R is a differential operator which is obtained from P and Q by a procedure similar to Euclid's algorithm for determining the greatest common divisor of two polynomials. If n and m are the orders of P and Q respectively with $n > m$, the order of R is $n - m$. If there exists no equation of order less than m over the same coefficient domain which has its solution in common with Q, the latter is called an *irreducible factor* of P. $R(y) = 0$ may be reducible as well and a corresponding decomposition may be obtained in the same way as for $P(y)$. This process is continued until the decomposition

$$P = Q_\lambda Q_{\lambda-1} \cdots Q_2 Q_1$$

into irreducible components is obtained. It is not unique. The arbitrariness involved is described by the following fundamental theorem due to Landau [1]. In any two decompositions of the differential equation $P(y) = 0$ into irreducible components, the number of factors and its orders are the same up to permutations. There is a one-to-one correspondence between pairs of factors from either decomposition such that both are of the same kind. This restriction does not exclude the possibility that there are equations which allow infinitely many decompositions into irreducible components.

If a differential equation is irreducible, its differential Galois group is transitive. From a theoretical point of view this is an important constraint and has its obvious counterpart in the Galois theory of algebraic equations. If an equation is reducible, its decomposition into irreducible components is usually an important step towards finding its solutions. Assume that a decomposition (1) has been obtained and that $\{y_1, \ldots, y_m\}$ is a fundamental system for $Q(y) = 0$. Obviously it also solves $P(y) = 0$. To obtain the remaining elements of a fundamental system for this latter equation,