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Canonical form transformation of second order differential equations with Lie symmetries

Fritz Schwarz

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Abstract. The most difficult part of solving an ordinary differential equation (ode) by Lie's symmetry theory consists of transforming it into a canonical form corresponding to its symmetry type. In this article, for all possible symmetry types of a quasilinear second order ode, theorems are obtained that reduce the transformation into canonical form to solving linear partial differential equations (pde's) or certain Riccati equations. They allow it to determine *algorithmically* the finite transformation functions to canonical form that are Liouvillian over the base field of the given ode. The knowledge of the infinitesimal symmetry generators is *not* required. Fundamental new concepts that are applied are the Janet base of a system of linear pde's and its decomposition into completely reducible components, i. e. the analogue to Loewy's decomposition of linear ode's.

1. Introduction

The most powerful methods for obtaining closed form solutions of nonlinear ordinary differential equations (ode's) are based on Lie's symmetry theory. Yet for more than fifty years after his death it has virtually never been applied for solving practical problems, only during the last decade some activity in this area has emerged. The collection of solved equations by Kamke [1] for example does not even mention his name, although almost all solutions given there are the consequence of a symmetry. This is essentially due to two reasons. On the one hand, for any nontrivial example the amount of calculations necessary for applying Lie's theory makes it impossible to be performed by pencil and paper. Secondly the theory as described by Lie does not allow it to design solution algorithms in a straightforward manner because various parts of it are not constructive.

After Lie had recognized that the symmetry of an ode is the fundamental new concept for finding its solutions in closed form, he has described essentially two versions of a solution procedure based on it. Originally he applied the symmetry

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