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Rational Solutions of Riccati-like Partial Differential Equations

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When factoring linear partial differential systems with a finite-dimensional solution space or analysing symmetries of nonlinear ODEs, we need to look for rational solutions of certain nonlinear PDEs. The nonlinear PDEs are called Riccati-like because they arise in a similar way as Riccati ODEs. In this paper we describe the structure of rational solutions of a Riccati-like system, and an algorithm for computing them. The algorithm is also applicable to finding all rational solutions of Lie's system $\{\partial_x u + u^2 + a_1 u + a_2 v + a_3,\ \partial_y u + uv + b_1 u + b_2 v + b_3,\ \partial_x v + uv + c_1 u + c_2 v + c_3,\ \partial_y v + v^2 + d_1 u + d_2 v + d_3\}$, where $a_1, \ldots, d_3$ are rational functions of x and y.

1. Introduction

Riccati's equation is one of the first examples of a nonlinear differential equation that was considered extensively in the literature, shortly after Leibniz and Newton introduced the concept of the derivative of a function at the end of the 17th century. Riccati equations occur in many problems of mathematical physics and pure mathematics. A good survey is given in the book by Reid (1972).

Of particular importance is the relation between Riccati's equation and a linear ODE $y'' + ay' + by = 0$ with $a, b \in \mathbb{C}(x)$, where $\mathbb{C}$ is the field of the complex numbers. For example, solutions h with the property that the quotient $p = h'/h \in \mathbb{C}(x)$ may be represented as $h = \exp\left(\int p\, dx\right)$ if p satisfies the first-order Riccati equation $p' + p^2 + ap + b = 0$. Equivalently, this linear ODE allows the first-order right factor $y' - qy$ over $\mathbb{C}(x)$ if q obeys the same equation as p. In general, finding the first-order right rational factors of a linear homogeneous ODE is equivalent to finding the rational solutions of its associated Riccati equation (see, for example, Singer, 1991).

It turns out that this correspondence carries over to systems of linear homogeneous partial differential equations with a finite-dimensional solution space. Systems of this kind occur in Lie's symmetry theory for solving nonlinear ODEs and related equivalence problems. For example, Lie studied the coherent nonlinear system

$$\begin{aligned}\{\partial_x u + u^2 + a_1 u + a_2 v + a_3, \quad & \partial_y u + uv + b_1 u + b_2 v + b_3, \\ \partial_x v + uv + c_1 u + c_2 v + c_3, \quad & \partial_y v + v^2 + d_1 u + d_2 v + d_3\}\end{aligned} \tag{1.1}$$

for the first time in connection with the symmetry analysis of second-order ODEs with projective symmetry group (see, for example, Lie 1873, p. 365). It is suggested therefore

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