Symmetries of Second- and Third-Order Ordinary Differential Equations

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Abstract. In order to apply Lie's symmetry theory for solving a differential equation it must be possible to identify the group of symmetries leaving the equation invariant. The answer is obtained in two steps. At first a classification of the possible symmetries of equations of the respective order is determined. Secondly a decision procedure is provided which allows to identify the symmetry type within this classification. For second-order equations the answer has been obtained by Lie himself. In this article the complete answer for quasilinear equations of order three is given. An important tool is the Janet base representation for the determining system of the symmetries.

1 Introduction

In general there are no solution algorithms available for solving nonlinear ordinary differential equations (ode's) in closed form. Usually several heuristics are applied in order to find a solution without any guarantee however to find one, or even to assure its existence. Stimulated by Galois' ideas for solving algebraic equations, Sophus Lie got interested in these problems around the middle of the 19^{th} century. His most significant recognition was that the transformation properties of an ode under certain groups of continuous transformations play a fundamental role for answering this question, very much like the permutations of the solutions of an algebraic equation furnish the key to understanding its solution behavior.

With the insight gained by analysing simple examples of solvable equations, Lie developed a solution scheme for ode's that may be traced back to the following question: Are there any transformations leaving the form of the given equation invariant? If the answer is affirmative these transformations are called its *symmetries.* In general, equations that may be transformed into each other are called *equivalent*, they form an *equivalence class* and share the same *symmetry type.* The union of all equivalence classes sharing the same symmetry type is called a *symmetry class.* If the solution of a canonical representative within an equivalence class may be obtained, and moreover it is possible to determine the transformation of the given equation to this canonical form, the original problem is solved.

* Supported in part by INTAS Grant 99-0167

F. Winkler and U. Langer (Eds.): SNSC 2001, LNCS 2630, pp. 108–139, 2003.