Janet Bases for Symmetry Groups

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1 Introduction

The subject of this article are systems of linear homogeneous partial differential equations (pde's) of various kinds. Above all such equations are characterized by the number m of dependent and the number n of independent variables. Additional quantities of interest are the number of equations, the order of the highest derivatives that may occur and the function field in which the coefficients are contained. Without further specification it is the field of rational functions in the independent variables. The basic new concept to be considered in this article is the *Janet base*. This term is chosen because the French mathematician Maurice Janet (Janet 1920) recognized its importance and described an algorithm for obtaining it. After it had been forgotten for about fifty years, it was rediscovered (Schwarz 1992) and utilized in various applications as it is described later on.

The theory of systems of linear homogeneous pde's is of interest for its own right, independent of its applications e. g. for finding symmetries and invariants of differential equations. Any such system may be written in infinitely many ways by linearly combining its members or derivatives thereof without changing its solution set. In general it is a difficult question whether there exist nontrivial solutions at all, or what the degree of arbitrariness of the general solution is. It may be a set of constants, or a set of functions depending on a differing number of arguments. This problem was the starting point for Janet. A *Janet base* is a unique representation of such systems of pde's that provides important information on its solutions similar to a Gröbner base representation of a system of algebraic equations, (Buchberger 1970) and (Buchberger 1985). Both a Gröbner base and a Janet base have one important feature in common: Except in very simple cases it is virtually impossible to calculate any of them by pencil-and-paper, i. e. an efficient computer algebra implementation is crucial for utilizing it in practical problems.

Basic concepts in Janet's theory are *terms* and *term orderings.* A term $t_{i,j}$ in a system of linear homogeneous pde's is either a dependent variable or a derivative of it. A *term ordering* is a linear order that is consistent with taking derivatives. This means a relation $t_{i,j} > t_{i',j'}$ remains valid if the same derivative ∂ is applied to either side of the order relation. Any linear homogeneous system of pde's in this article is arranged in the form