On Discrete Subgroups of the Lorentz Group.

F. SCHWARZ

Universität Kaiserslautern, Fachbereich Physik
675 Kaiserslautern, Postfach 3049, Deutschland

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It is probably one of the most fundamental principles in theoretical physics that physical systems are invariant under Lorentz transformations. Consequently the underlying invariance group, *i.e.* the Lorentz group, has most intensively been studied by physicists (1) and also by mathematicians (2). So the properties of this group seem to be completely known. However, there is one aspect which does not seem to have attracted attention adequately until now, *i.e.* the incomplete knowledge of the discrete subgroups of the Lorentz group. There are two types of discrete subgroups which have been well known to physicists for a long time. In the first place these are the representatives of the three cosets of the full Lorentz group $O_{3,1}$ with respect to the identity component $SO_{0(3,1)}$. In physical terms these are the space inversion, the time inversion and the combination of these two operations, together with the unit element. Secondly there are the discrete subgroups of the three-dimensional rotation group which are the cornerstone for the classification of crystal systems. All these groups have a fundamental meaning in theoretical physics. So it does not seem to be unreasonable to ask whether there exists still more discrete subgroups of the Lorentz group and how to find them.

It is the purpose of this note to point out that a partial answer to this question can be obtained from the theory of automorphic functions of one complex variable. The basic work in this field has already been completed almost one hundred years ago by FRICKE and KLEIN (3). Some more recent publications in this field are the books by FORD (4), LEHNER (5) and GELFAND *et al.* (6). For all definitions and results used subsequently we refer the reader to these books.

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(1) E. P. WIGNER: *Ann. of Math.*, **40**, 149 (1939).

(2) I. M. GELFAND, R. A. MINLOS and Z. YA. SHAPIRO: *Representations of the Rotation and Lorentz Groups and Their Applications* (Oxford and New York, N. Y., 1963); M. A. NAIMARK: *Linear Representations of the Lorentz Group* (London, 1964). These are two standard references on this subject. It is not our intention to give a complete list of the relevant literature which would probably fill several dozens of pages.

(3) R. FRICKE and F. KLEIN: *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Vol. **1** (Leipzig, 1890); Vol. **2** (Leipzig, 1892). *Vorlesungen über die Theorie der automorphen Funktionen*, Vol. **1** (Leipzig, 1897); Vol. **2**, part 1 (Leipzig, 1901); part 2 (Leipzig, 1912). These books have been reprinted by the Johnson Reprint Corporation (New York, N. Y., 1966).

(4) L. R. FORD: *Automorphic Functions* (New York, N. Y., 1972).

(5) J. LEHNER: *A Short Course in Automorphic Functions* (New York, N. Y., 1966).

(6) I. M. GELFAND, M. I. GRAEV and I. I. PYATETSKII-SHAPIRO: *Representation Theory and Automorphic Functions* (Philadelphia, Pa., 1966).

The theory of automorphic functions consists essentially in studying those functions of a complex variable z which are invariant under certain discrete groups of linear transformations $T: z \to z'$ which have the form $z' = (\alpha z + \beta)/(\gamma z + \delta)$. So as a first step one has to study the discrete groups of linear transformations. The second step deals with the functions which are invariant under the respective group, *i.e.* the corresponding automorphic functions. It is the first step which is of interest for our present work because the coefficients of these transformations are restricted by $\alpha\beta - \gamma\delta = 1$. This means that the matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is an element of the group $SL_{2,C}$. There exists a homomorphism between this group and the identity component $SO_{0(3,1)}$ of the homogeneous Lorentz group. So to any subgroup of $SL_{2,C}$ there corresponds a subgroup of $SO_{0(3,1)}$. This homomorphism will be exploited to determine several classes of discrete subgroups of $SO_{0(3,1)}$ and to study some elementary properties of them.

There are three major classes of discrete subgroups of $SL_{2,C}$:

1) Elementary groups. They consist of the finite groups and the groups with one or two limit points.

2) Fuchsian groups. A group is called Fuchsian if its transformations have a common fixed circle and if each transformation carries the interior of the fixed circle into itself.

3) Kleinian groups. A group is called Kleinian if it does not belong to one of the preceding classes.

The knowledge about the Kleinian groups is very incomplete until now and so we do not consider them any longer.

The elementary groups are completely known. The finite groups do not contain any Lorentz transformations. They correspond to the crystallographic groups and are omitted from our discussion. The groups with one limit point consist of the simply and doubly periodic groups. The former class contains the matrices

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}1 & n\omega\\ 0 & 1\end{pmatrix}. \tag{1}$$

The parameter $\omega = \varrho \exp[i\delta]$, ϱ and δ real, $0 \leqslant \varrho < \infty$, $0 \leqslant \delta < 2\pi$, determines the group, the integer n the group elements within a given group. The corresponding matrix $L \in SO_{0(3,1)}$ has the elements [7]

$$\{L_{ij}\} = \begin{pmatrix} 1 & 0 & -n\varrho\cos\delta & n\varrho\cos\delta \\ 0 & 1 & -n\varrho\sin\delta & n\varrho\sin\delta \\ n\varrho\cos\delta & n\varrho\sin\delta & 1-\frac{1}{2}n^2\varrho^2 & \frac{1}{2}n^2\varrho^2 \\ n\varrho\cos\delta & n\varrho\sin\delta & -\frac{1}{2}n^2\varrho^2 & 1+\frac{1}{2}n^2\varrho^2 \end{pmatrix}. \tag{2}$$

(7) A four-vector has the components (x_1, x_2, x_3, x_4). We use the metric $x_1^2+x_2^2+x_3^2-x_4^2$ in agreement with the book by NAIMARK which is quoted under ref. (8).

The doubly periodic groups consist of the elements

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\begin{pmatrix}1 & n_1\omega_1+n_2\omega_2\\ 0 & 1\end{pmatrix}. \tag{3}$$

In this case there are four real parameters determining the group, *i.e.* $\omega_k = \varrho_k \exp[i\delta_k]$, $k = 1, 2$; they obey the same restrictions as ϱ and δ for the simply periodic groups. A group element is determined by the two integers n_1 and n_2. If one defines the functions

$$f = f(n_k, \omega_k) = n_1\varrho_1 \cos\delta_1 + n_2\omega_2 \cos\delta_2\,, \tag{4a}$$

$$g = g(n_k, \omega_k) = n_1\varrho_1 \sin\delta_1 + n_2\omega_2 \sin\delta_2\,, \tag{4b}$$

$$h = h(n_k, \omega_k) = \tfrac{1}{2}[n_1^2\varrho_1^2 + n_2\varrho_2^2 + 2n_1n_2\varrho_1\varrho_2\cos(\delta_1-\delta_2)]\,, \tag{4c}$$

the corresponding Lorentz matrix may be written as

$$\{L_{ij}\} = \begin{pmatrix}1 & 0 & -f & f\\ 0 & 1 & -g & g\\ f & g & 1-h & h\\ f & g & -h & 1+h\end{pmatrix}. \tag{5}$$

Finally they are the groups with two limit points. Their elements are the matrices of the form

$$\{L_{ij}\} = \begin{pmatrix}K^{n/2}K_1^{m/2} & 0\\ 0 & K^{-n/2}K_1^{-m/2}\end{pmatrix}, \tag{6a}$$

$$\{L_{ij}\} = \begin{pmatrix}0 & iK^{n/2}K_1^{m/2}\\ iK^{-n/2}K_1^{-m/2} & 0\end{pmatrix}. \tag{6b}$$

The parameters $K = \varrho\exp[i\delta] = \exp[\sigma]\exp[i\delta]$, ϱ, σ and δ real, $0 \leqslant \varrho < \infty$, $-\infty < \sigma < \infty$ ($\varrho \neq 1$, $\sigma \neq 0$), $0 \leqslant \delta < 2\pi$, and $K_1 = \exp[2\pi i/k]$, $k = 1, 2, \ldots$, determine the group. The integers $n = 0, \pm 1, \pm 2, \ldots$ and $m = 0, 1, 2, \ldots, k-1$ fix a group element within a group. The matrices (6*a*) form a group by themselves which contains the subgroups of elements with $k = 1$. The matrices of $SO_{0(3,1)}$ corresponding to the matrices (6) of $SL_{2,C}$ are given through

$$\{L_{ij}\} = \begin{pmatrix}\cos\left(n\delta + \dfrac{2\pi m}{k}\right) & \pm\sin\left(n\delta + \dfrac{2\pi m}{k}\right) & 0 & 0\\ \sin\left(n\delta + \dfrac{2\pi m}{k}\right) & \pm\cos\left(n\delta + \dfrac{2\pi m}{k}\right) & 0 & 0\\ 0 & 0 & \pm\cosh n\sigma & \sinh n\sigma\\ 0 & 0 & \pm\sinh n\sigma & \cosh n\sigma\end{pmatrix}. \tag{7}$$

The signs in this equation are correlated. The upper and lower signs correspond to the matrices (6*a*) and (6*b*) respectively. For $k=1$ and $\delta=0$ the transformations (7) represent the simplest example of a discrete subgroup of a hyperbolic group, *i.e.* discrete Lorentz boosts in the 3-direction. If only $\delta=0$, the group (7) contains in addition the cyclic group of order k in the (1-2)-plane.

Unlike the case of the elementary groups, the Fuchsian groups are not completely known. Its most important representative is the modular group. It consists of those matrices of $SL_{2,C}$ the elements of which are real integers, *i.e.* a two-by-two unimodular matrix belongs to it if

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}k & l\\ m & n\end{pmatrix} \tag{8}$$

with $k, l, m, n = 0, \pm 1, \pm 2, \ldots$ and $kn-lm=1$. This implies that the corresponding element from $SO_{0(3,1)}$ leaves one axis unchanged because it belongs actually to a subgroup $SO_{0(2,1)}\subset SO_{0(3,1)}$. This may also be seen from the fact that the elements of the form (8) belong to $SL_{2,R}$ which is isomorphic to $SO_{0(2,1)}$. If we define the four quantities p, q, r and s through

$$p=\tfrac{1}{2}(k^2-l^2-m^2+n^2)\,, \tag{9a}$$

$$q=\tfrac{1}{2}(k^2+l^2-m^2-n^2)\,, \tag{9b}$$

$$r=\tfrac{1}{2}(k^2-l^2+m^2-n^2)\,, \tag{9c}$$

$$s=\tfrac{1}{2}(k^2+l^2+m^2+n^2)\,, \tag{9d}$$

the matrix of $SO_{0(3,1)}$ corresponding to the element (8) of $SL_{2,C}$ may be written as

$$\{L_{ij}\}=\begin{pmatrix}kn+lm & 0 & km-ln & km+ln\\ 0 & 1 & 0 & 0\\ kl-mn & 0 & p & q\\ kl+mn & 0 & r & s\end{pmatrix}. \tag{10}$$

The structure of the modular group is quite complicated. Until now the lattice of its subgroups is only partially known. In fact, the most complete treatment of the modular group is still the work by FRICKE and KLEIN [3], see especially the first book quoted under this reference. In this paper we restrict our discussion to the best-known subgroups of the modular group. These are the principal congruence groups of level N. They consist of those matrices of the modular group which are congruent to the identity modulo N, *i.e.* a matrix belongs to it if

$$\begin{pmatrix}k & l\\ m & n\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ (\mathrm{mod}\ N)\,. \tag{11}$$

The matrices with this property may be written in the form $\begin{pmatrix}Nk+1 & Nl\\ Nm & Nn+1\end{pmatrix}$ with $k, l, m, n = 0, \pm 1, \pm 2, \ldots$ and $N[N(kn-lm)+k+n]=0$. N is a positive in-

TABLE 1. – *The elementary groups with the exception of the finite groups.* The spacelike, lightlike and timelike lattices are generated from the starting vectors $(x_1, x_2, x_3, x_0) = (1, 0, 0, 0)$, $(1, 0, 0, 1)$ and $(0, 0, 0, 1)$ respectively. The functions f, g and h occurring in the description of the doubly periodic groups are defined by eqs. (4).

Type of corresponding subgroup of $SL_{2,c}$	Parameters determining the group	Parameters determining the group element	Lattice in Minkowski space generated by the respective group: Spacelike	Lightlike	Timelike
Simply periodic groups	$\omega = \varrho \exp[i\delta]$ ϱ, δ real	$n = 0, \pm 1, \pm 2, \ldots$	$\begin{pmatrix} 1 \\ 0 \\ n\varrho\cos\delta \\ n\varrho\cos\delta \end{pmatrix}$	$\begin{pmatrix} 1 + n\varrho\cos\delta \\ n\varrho\sin\delta \\ n\varrho\cos\delta + \frac{1}{2}n^2\varrho^2 \\ n\varrho\cos\delta + \frac{1}{2}n^2\varrho^2 + 1 \end{pmatrix}$	$\begin{pmatrix} n\varrho\cos\delta \\ n\varrho\sin\delta \\ \frac{1}{2}n^2\varrho^2 \\ 1 + \frac{1}{2}n^2\varrho^2 \end{pmatrix}$
Doubly periodic groups	$\omega = \varrho_k \exp[i\delta_k]$ ϱ_k, δ_k real $k = 1, 2$	$n_k = 0, \pm 1, \pm 2, \ldots$ $k = 1, 2$	$\begin{pmatrix} 1 \\ 0 \\ f \\ f \end{pmatrix}$	$\begin{pmatrix} 1 + f \\ g \\ f + h \\ f + h + 1 \end{pmatrix}$	$\begin{pmatrix} f \\ g \\ h \\ 1 + h \end{pmatrix}$
Groups with two limit points	$K = \varrho \exp[i\delta] = \exp[\sigma]\exp[i\delta]$ ϱ, σ, δ real $\varrho \neq 1$, $\sigma \neq 0$, $K_1 = \exp\left[\frac{2\pi i}{k}\right]$, $k = 1, 2, \ldots$	$n = 0, \pm 1, \pm 2 \ldots$ $m = 0, 1, \ldots, k-1$	$\begin{pmatrix} \cos\left(n\delta + \frac{2\pi m}{k}\right) \\ \sin\left(n\delta + \frac{2\pi m}{k}\right) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \cos\left(n\varrho + \frac{2\pi m}{k}\right) \\ \sin\left(n\varrho + \frac{2\pi m}{k}\right) \\ \sinh n\sigma \\ \cosh n\sigma \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \sinh n\sigma \\ \cosh n\sigma \end{pmatrix}$

TABLE II. – *Some properties of the modular group and the principal congruence subgroup of level N*. The quantities p, q, r and s are defined by eqs. (9).

Type of corresponding subgroup of $SL_{2,C}$	Parameters determining the group	Parameters determining a group element
Modular group	none	$k, l, m, n = 0, \pm 1, \pm 2, \ldots$ $kn - lm = 1$
Principal congruence group of level N	$N = 1, 2, \ldots$	$k, l, m, n = 0, \pm 1, \pm 2, \ldots$ $N(kn - lm) + k + n = 0$

teger. The corresponding matrix from $SO_{0(3,1)}$ is given through

(12) $\{L_{ij}\} =$

$$= \begin{pmatrix} N[N(kn+lm)+k+n]+1 & 0 & N[N(km-ln)+m-l] & N[N(km+ln)+m+l] \\ 0 & 1 & 0 & 0 \\ N[N(kl-mn)+l-n] & 0 & N^2p+N(k+n)+1 & N^2q+N(k-n) \\ N[N(kl+mn)+l+n] & 0 & N^2r+N(k-n) & N^2s+N(k+n)+1 \end{pmatrix}.$$

One of the most interesting properties of the discrete subgroups of $SO_{0(3,1)}$ is the fact that each of them generates a characteristic lattice in Minkowski space if its elements are applied to three suitably chosen starting vectors. These lattices are invariant under the respective groups and may be regarded as the relativistic generalization of the prisms, cubes, tetrahedrons etc. which are invariant under the discrete subgroups of the proper rotation group.

In tables I and II the lattice vectors generated by the elementary and some Fuchsian groups respectively are given. To get an idea of how these lattices in Minkowski space may look like, we have explicitly calculated them on a computer for two cases. The result is plotted in fig. 1. The starting vectors for both cases are (1, 0, 0, 0), (0, 0, 0, 1) and (1, 0, 0, 1). The transformations (10) and (12) for $N = 2$ are applied to these vectors. The $(x_0\text{-}|\boldsymbol{x}|)$-plot in fig. 1 is restricted to the range $|\boldsymbol{x}| < 5$. Clearly the complete lattice is infinite. Due to the fact that only the absolute value of the space component is shown, the structure of the lattices is partly lost. Nevertheless we think that this figure may help to get an idea of their shape.

contained in it. The starting vectors for the lattices in Minkowski space are the same as before.

Lattice in Minkowski space generated by the respective group		
Spacelike	Lightlike	Timelike
$\begin{pmatrix} kn+lm \\ 0 \\ kl-mn \\ kl+mn \end{pmatrix}$	$\begin{pmatrix} kn+lm+km+ln \\ 0 \\ \frac{1}{2}[(k+l)^2-(m+n)^2] \\ \frac{1}{2}[(k+l)^2+(m+n)^2] \end{pmatrix}$	$\begin{pmatrix} km+ln \\ 0 \\ q \\ s \end{pmatrix}$
$\begin{pmatrix} N[N(kn+lm)+k+n]+1 \\ 0 \\ N[N(kl-mn)+l-m] \\ N[N(kl+mn)+l+m] \end{pmatrix}$	$\begin{pmatrix} N[N(k+l)(m+n)+k+l+m+n]+1 \\ 0 \\ \frac{1}{2}N^2[(k+l)^2-(m+n)^2]+N(k+l-m-n) \\ \frac{1}{2}N^2[(k+l)^2+(m+n)^2]+N(k+l+m+n) \end{pmatrix}$	$\begin{pmatrix} N[N(km+ln)+l+m] \\ 0 \\ N^2q+N(k-n) \\ N^2s+N(k+n)+1 \end{pmatrix}$

With the discussion of the modular group we have essentially exhausted the possibility to determine discrete subgroups of Lorentz transformations by using results from the theory of automorphic functions.

There remains the question how to continue our investigations. On the one hand it would be very interesting to determine additional discrete subgroups of $SO_{0(3,1)}$ and possibly prove the completeness of the resulting list of subgroups. As a preliminary result one could try to obtain the discrete subgroups of $SO_{0(2,1)}$. It might be possible to find some « deforming process » as it is used for example for determining the irreducible representations of certain Lie algebras. In this way one could take advantage of the fact that the discrete subgroups of the compact groups SO_3 and SO_4 are completely known [8]. Further it would be interesting to extend these results from the identity component to the full groups which consist of four disconnected pieces. So far we have only discussed the homogeneous groups. Clearly there arises the question as to whether there exist also discrete subgroups of the Poincaré group, and if so what they look like. This amounts to extending the discrete subgroups of the homogeneous groups by translations along the co-ordinate axes. The general structure of these groups seems to be known [9].

Above all, however, there remains the central question whether the discrete subgroups of the Lorentz group have anything to do with physics. One might think of a classification of the discrete energy states of matter in a similar way as the crystal structures occurring in nature are classified according to the space groups. For this purpose the lattices in Minkowski space should be related somehow to the energy-momentum four-vectors representing elementary particles. Maybe this is not possible. In this case this work would be only a mathematical exercise. On the other hand it could be that our

(8) J. Neubüser, H. Wondratschek and R. Bülow: *Acta Cryst.*, A **27**, 517, 520, 523 (1971).
(9) E. Ascher and A. Janner: *Helv. Phys. Acta*, **38**, 551 (1965); *Comm. Math. Phys.*, **11**, 138 (1968).

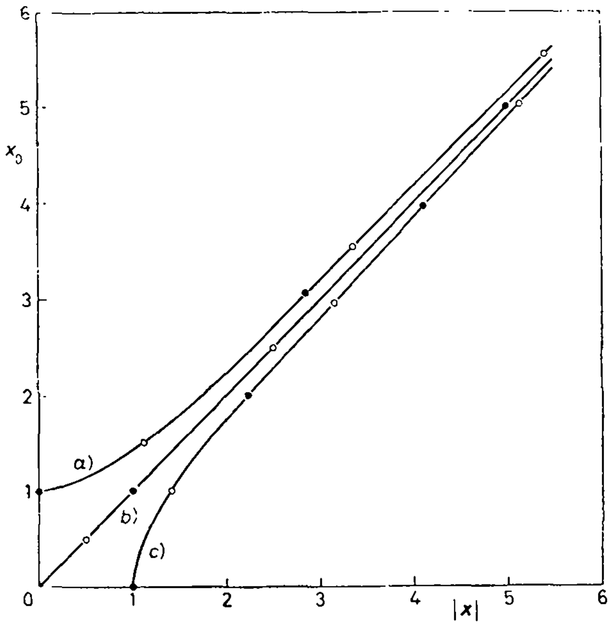

Fig. 1. – Two examples for the lattices in Minkowski space generated by discrete subgroups of the Lorentz group. The starting vectors (x_1, x_2, x_3, x_0) are (1, 0, 0, 0), (0, 0, 0, 1) and (1, 0, 0, 1) for the spacelike, timelike and lightlike lattice respectively. The figure is restricted to the region $|x|<5$. The complete lattices contain an infinite number of points. The open circles and the dots together represent the lattice generated by the modular group. The dots alone are generated through the principal congruence group of level 2. *a*) $|x|^2 - x_0^2 = -1$, *b*) light-cone, *c*) $|x|^2 - x_0^2 = 1$.

knowledge of the discrete subgroups of the Lorentz group is too poor until now so that the connection with physics cannot be seen yet. If this is true it would certainly be a rewarding task to continue along the lines described above.