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THE RIQUIER–JANET THEORY AND ITS APPLICATION TO NONLINEAR EVOLUTION EQUATIONS

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It is shown that by taking into account all integrability conditions for a given system of partial differential equations a more systematic treatment of several problems in the theory of nonlinear evolution equations may be performed. The basic prerequisites are the theory of Riquier and Janet of partial differential equations and the application of the computer algebra system REDUCE. Several kinds of Bäcklund transformations and the prolongation of Estabrook and Wahlquist are discussed in detail.

1. Introduction

Ever since the discovery of the remarkable phenomenon by Zabusky and Kruskal [1] which they termed "recurrence of the initial state" the research activity related to nonlinear evolution equations has increased tremendously. A completely new area in mathematical physics centered around the so-called solitons has arisen. Several old ideas like solitary wave solutions and Bäcklund transformations have merged together with completely new concepts, some of the most important of which are the inverse spectral transform and the method of Estabrook and Wahlquist. Many of these ideas are related to each other in an intricate way but it does not seem clear to date how exactly they are connected. For a rather complete review on this subject and references to the original literature we refer the reader to the book on solitons edited by Bullough and Caudrey [2].

At a superficial view, several of these ideas have been invented somewhat accidentally although this is not really the case. The initialization of the subject by Zabusky and Kruskal [1] for example has been the result of an extensive research on the so called Fermi–Pasta–Ulam problem in statistical mechanics. Nevertheless, a systematic and straightforward method for arriving at many of the important results in this field does not seem to exist. This is true to a lesser extent for the method of Estabrook and Wahlquist [3] which has been mentioned above. The essence of their prolongation procedure for a given partial differential equation is to introduce new dependent variables, also defined by differential equations, in such a way that the integrability condition for the existence of the new variables is precisely the equation originally given [4]. The success of this idea shows the fundamental importance of taking into account appropriately the integrability conditions for any system of partial differential equations.

There is a rather complete theory on this subject for systems of partial differential equations of first order for a single unknown function [5]. The integrability conditions for such systems may be expressed in terms of Poisson or Jacobi brackets. By adding certain nonvanishing brackets to a given system, it may always be achieved that the extended system is completely integrable. For systems of higher order or, what amounts to the same thing, systems containing more than one unknown function, the answer is more complicated. It does

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