SIAM REVIEW
Vol. 30, No. 3, September 1988

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SYMMETRIES OF DIFFERENTIAL EQUATIONS: FROM SOPHUS LIE TO COMPUTER ALGEBRA*

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Abstract. The topic of this article is the symmetry analysis of differential equations and the applications of computer algebra to the extensive analytical calculations which are usually involved in it. The whole area naturally decomposes into two parts depending on whether ordinary or partial differential equations are considered. We show how a symmetry may be applied to lower the order of an ordinary differential equation or to obtain similarity solutions of partial differential equations. The computer algebra packages SODE and SPDE, respectively, which have been developed to perform almost all algebraic manipulations necessary to determine the symmetry group of a given differential equation, are presented. Furthermore it is argued that the application of computer algebra systems has qualitatively changed this area of applied mathematics.

Key words. differential equations, symmetries, computer algebra

AMS(MOS) subject classifications. 3504, 35A99

1. Introduction. During the last two decades a change has occurred in applied mathematics that is probably even more severe than the introduction of computers for performing numerical calculations about forty years ago. That means that large computers have rendered it feasible to perform analytical calculations automatically as well. Although the idea of mechanizing analytical calculations is already more than 100 years old—Charles Babbage was apparently the first person who carried it out by constructing his so-called analytical engine [23]—it took about 150 years after that until the large computer algebra systems for electronic computers came to be at our disposal.

The most important general-purpose computer algebra systems available today are MACSYMA by the Mathlab Group at MIT, REDUCE by A. C. Hearn at the Rand Corporation, MAPLE by B. Char at Waterloo, μ-Math by D. R. Stoutemyer of "The Softwarehouse" in Honolulu, SMP by S. Wolfram and SCRATCHPAD II by R. D. Jenks and D. Yun at IBM. Typically a computer algebra system provides modules for performing basic operations like simplification, differentiation, integration, factorization, etc. These algorithms are the building blocks for any other packages which may be developed by the user for special applications.

The availability of these computer algebra systems has a particularly strong influence on those areas of applied mathematics where large analytical manipulations are necessary for obtaining a certain result. Applying a computer algebra system means to become accustomed to a completely new working style. Pencil and paper work is almost completely eliminated. Instead of working out a problem in the old-fashioned way over and over again with varying input, a solution strategy by applying computer algebra methods is developed, the main steps of which may be described as follows:

(1) Identify a sufficiently general class of problems and decompose it into subproblems which must be manageable and as independent of each other as possible.

* Received by the editors June 16, 1986; accepted for publication (in revised form) February 15, 1987.

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